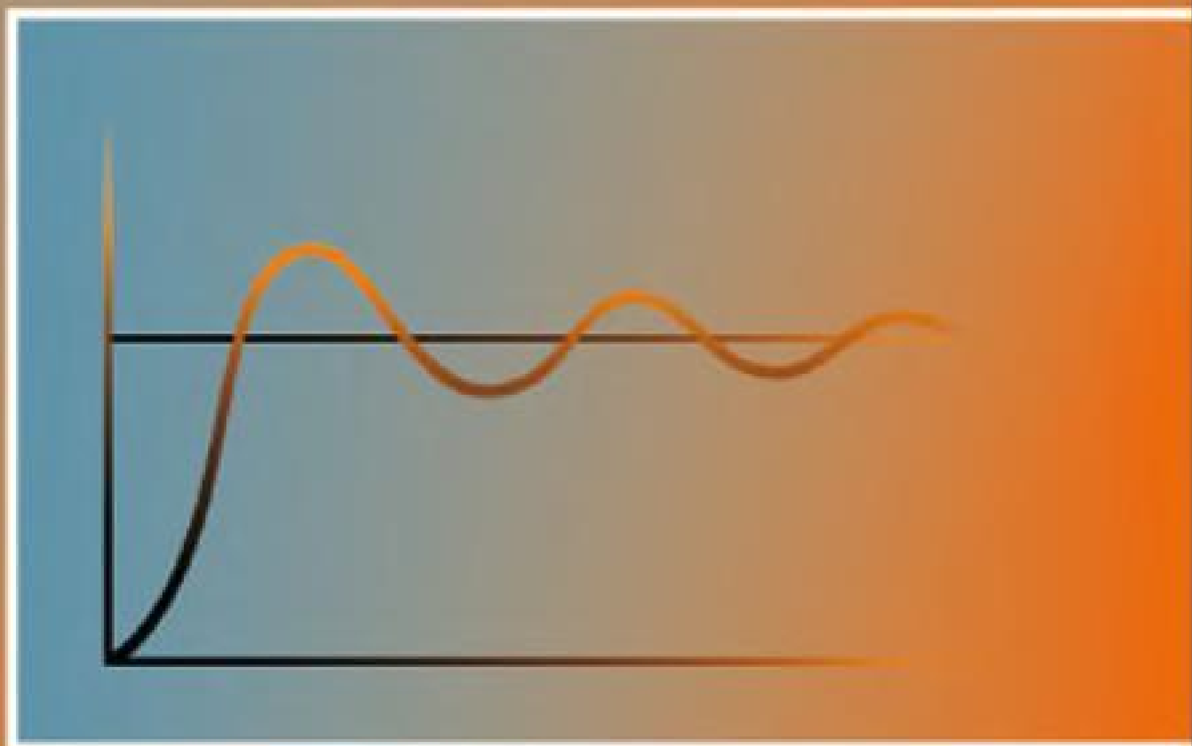
Eastern Economy Edition

Control Systems



A. Anand Kumar

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The design and compensation of control systems can be carried out in time-domain or frequency-domain, but design and compensation in frequency-domain is much simpler. In frequency-domain, compensation using Bode plots is much simpler than using other plots and is discussed in Chapter 9.

State variable approach is the modern direct, time-domain approach for the analysis and design of control systems. Detailed analysis of systems using state variables as well as the controllability and observability of control systems are the topics discussed in Chapter 10.

A large number of examples have been worked out to help students understand the concepts. Extensive short questions and answers are given at the end of each chapter to enable the students to prepare for the examinations very thoroughly. Review questions, Fill in the blank type questions, objective type multiple choice questions and numerical problems are included at the end of each chapter to enable the students to build a clear understanding of the subject matter discussed in the text and also to assess their learning. The answers to all these are also given. Almost all the solved and unsolved problems presented in this book have been classroom tested.

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A. Anand Kumar

1

Introduction to Control Systems

1.1 INTRODUCTION

A system is a collection of objects (components) connected together to serve an objective, or a system is a combination of components that act together to perform an objective. A control system is that means by which any quantity of interest in a machine, mechanism, or other equipment is maintained or altered in accordance with a desired manner, or simply a control system is a system in which the output quantity is controlled by varying the input quantity.

A physical system is a collection of physical objects connected together to serve an objective. No physical system can be represented in its full physical intricacies and, therefore, idealizing assumptions are always made for the purpose of analysis and synthesis of systems. An idealized physical system is called a *physical model*. A physical system can be modelled in a number of ways depending upon the specific problem to be dealt with and the desired accuracy. Once a physical model of a physical system is obtained, the next step is to obtain a mathematical model, which is the mathematical representation of the physical model, through use of appropriate physical laws. Depending upon the choice of variables and the coordinate system, a given physical model may lead to different mathematical models. A control system may be modelled as a scalar differential equation describing the system or as a state variable vector-matrix differential equation. The particular mathematical model which gives a greater insight into the dynamic behaviour of the physical system is selected. When the mathematical model of a physical system is solved for various input conditions, the result represents the dynamic response of the system.

Mathematical models of most physical systems are characterized by differential equations. A mathematical model is linear, if the differential equation describing it has coefficients which are either function only of the independent variable or are constants. If the coefficients of the describing differential equation are functions of time (the independent variable), then the mathematical model is linear time-varying. On the other hand, if the coefficients of the describing differential equations are constants, the model is linear time-invariant.

The differential equations describing a linear time-invariant system can be reshaped into different forms for the convenience of analysis. For example, for transient response or frequency response analysis of single-input-single-output linear systems, the transfer function representation forms a useful model. On the other hand, when a system has multiple inputs and outputs, the vector-matrix notation may be more convenient. The mathematical model of a system having been obtained, the available mathematical tools can then be utilized for analysis or synthesis of the system.

Powerful mathematical tools like the Fourier and Laplace transforms are available for use in linear systems. Unfortunately, no physical system in nature is perfectly linear. Therefore, certain assumptions must always be made to get a linear model which is a compromise between the simplicity of the mathematical model and the accuracy of results obtained from it. However, it may not always be possible to obtain a valid linear model, for example, in the presence of a strong nonlinearity or in the presence of distributive effects which cannot be represented by lumped parameters.

1.2 CLASSIFICATION OF CONTROL SYSTEMS

Control systems may be classified in a number of ways depending on the purpose of classification.

1. Depending on the hierarchy, control systems may be classified as
   (a) Open-loop control systems
   (b) Closed-loop control systems
   (c) Optimal control systems
   (d) Adaptive control systems
   (e) Learning control systems
2. Depending on the presence of human being as a part of the control system, control systems may be classified as
   (a) Manually controlled systems
   (b) Automatic control systems
3. Depending on the presence of feedback, control systems may be classified as
   (a) Open-loop control systems
   (b) Closed-loop control systems or feedback control systems
4. According to the main purpose of the system, control systems may be classified as
   (a) Position control systems
   (b) Velocity control systems
   (c) Process control systems
   (d) Temperature control systems
   (e) Traffic control systems, etc.

Feedback control systems may be classified in a number of ways depending on the purpose of classification.

1. According to the method of analysis and design, control systems may be classified as linear control systems and nonlinear control systems.
2. Depending on whether the parameters of the system remain constant or vary with time, control systems may be classified as time-varying control systems or time-invariant control systems.
3. According to the types of signals used in the system, control systems may be classified as
   (a) Continuous-data control systems and discrete-data control systems
   (b) ac (modulated) control systems and dc (unmodulated) control systems.
4. Depending on the application, control systems may be classified as position control systems, velocity control systems, process control systems, traffic control systems, etc.
5. Depending on the number of inputs and outputs, control systems may be classified as single-input-single-output (SISO) control systems and multi-input-multi-output (MIMO) control systems. MIMO systems are also called *multivariable systems*.
6. Depending on the number of open-loop poles of the system transfer function present at the origin of the s-plane, control systems may by classified as
   (a) Type-0
   (b) Type-1
   (c) Type-2 etc. systems.
7. Depending on the order of the differential equation used to describe the system, control systems may be classified as first-order control systems, second-order control systems, etc.
8. Depending on the type of damping, control systems may be classified as
   (a) Undamped systems
   (b) Underdamped systems
   (c) Critically damped systems
   (d) Overdamped systems.

1.2.1 Open-Loop Control Systems

Those systems in which the output has no effect on the control action, i.e. on the input are called *open-loop control systems*. In other words, in an open-loop control system, the output is neither measured nor fed back for comparison with the input. Open-loop control systems are not feedback systems. Any control system that operates on a time basis is open-loop.

In any open-loop control system, the output is not compared with the reference input. Thus, to each reference input, there corresponds a fixed operating condition; as a result, the accuracy of the system depends on the calibration. In the presence of disturbances, an open-loop control system will not perform the desired task because when the output changes due to disturbances, it is not followed by changes in input to correct the output. In open-loop control systems, the changes in output are corrected by changing the input manually. Open-loop control systems can be used in practice only if the relationship between the input and the output is known and if there are neither internal nor external disturbances. One practical example of an open-loop control system is a washing machine—soaking, washing and rinsing in the washer operate on a time basis. The machine does not measure the output signal, i.e. the cleanliness of the clothes. A traffic control system that operates by means of signals on a time basis is another example of an open-loop control system. A room heater without any temperature sensing device is also an example of an open-loop control system. The general block diagram of an open-loop system is shown in Figure 1.1.

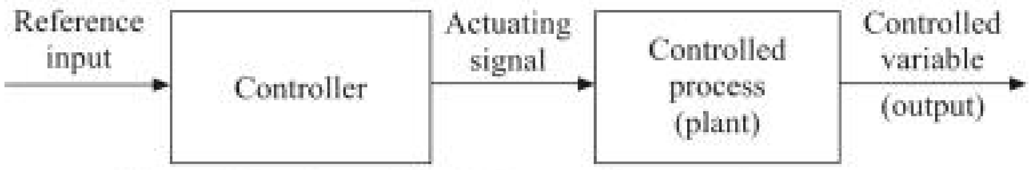


Figure 1.1 General block diagram of an open-loop system.

1.2.2 Closed-Loop Control Systems

Feedback control systems are often referred to as *closed-loop control systems*. In practice, the terms, 'closed-loop control' and 'feedback control' are used interchangeably. In a closed-loop control system, the actuating error signal which is the difference between the input signal and the feedback signal (which may be the output signal itself or a function of the output signal and its derivatives and/or integrals) is fed to the controller so as to reduce the error and bring the output of the system to a desired value. A system that maintains a prescribed relationship between the output and the reference input by comparing them and using the difference as a means of control is called a *feedback control system*. The term 'closed-loop control' always implies the use of feedback control action in order to reduce system error.

The general block diagram of an automatic control system is shown in Figure 1.2. It consists of an error detector, a controller, a plant and feedback path elements.

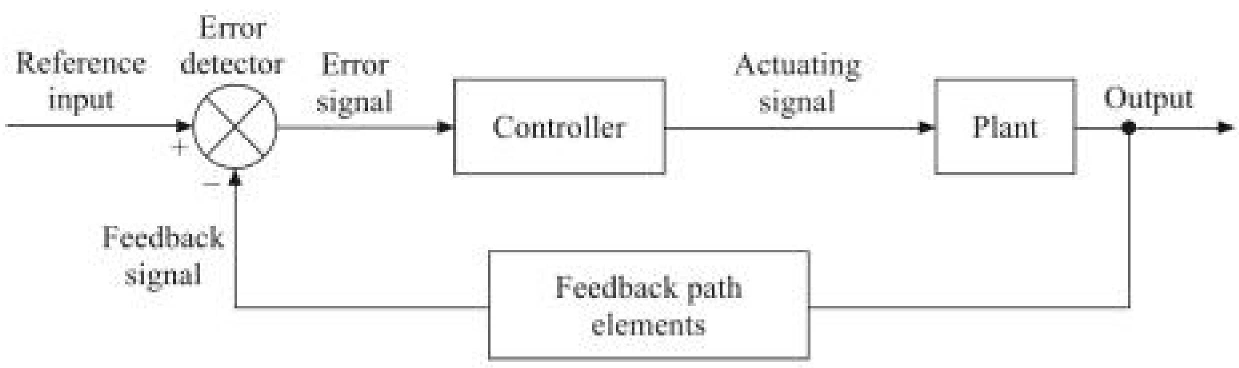


Figure 1.2 General block diagram of a closed-loop control system.

The reference input corresponds to desired output. The feedback path elements convert the output to a signal of the same type as that of the reference signal. The feedback signal is proportional to the output signal and is fed to the error detector. The error signal generated by the error detector is the difference between the reference signal and the feedback signal. The controller modifies and amplifies the error signal to produce better control action. The modified error signal is fed to the plant to correct its output.

An example of a feedback control system is a room temperature control system. By measuring the actual room temperature and comparing it with the reference temperature (desired temperature), the thermostat turns the heating or cooling equipment on or off in such a way as to ensure that the room temperature remains at a comfortable level regardless of outside conditions.

The following are some examples of control systems:

Traffic control system: Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals is based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of the traffic before giving the signals. It gives the signals in sequence as per the setting irrespective of the actual traffic. Since the time slots do not change according to traffic density, the system is an open-loop control system.

This open-loop traffic control system can be made as a closed-loop system if the time slots of the signals are based on the density of traffic. In a closed-loop traffic control system, the density of the traffic is measured on all the sides and the information is fed to a computer. The timings of the control signals are decided by the computer based on the density of traffic. Since the closed-loop system dynamically changes the timings, the flow of vehicles will be better than in that of an open-loop system.

Room heating system: A room heater without any temperature sensing device is an example of an open-loop control system. In this, an electric furnace is used to heat the room. The output is the desired room temperature. The temperature of the room is risen by the heat generated by the heating element. The output temperature depends on the time during which the supply to the heater remains ON. The ON-OFF time of the heater is set as per some calculation. Whatever may be the room temperature, after the set time, the heater will be OFF. The actual temperature is not compared with the reference temperature and the difference is not used for correction.

The above system becomes a closed-loop system if a thermostat is provided to measure the actual temperature, and the actual temperature is compared with the reference, and the difference is used to control the timing for which the heater is ON.

Washing machine: A washing machine without any cleanliness measuring system is an example of an open-loop control system. In this, the soaking, washing, and rinsing in the washer operate on a time basis. The machine ON time is set based on some calculation. The machine does not measure the output signal, that is the cleanliness of the clothes. Once the set ON time is over, the machine will automatically stop, whatever may be the level of cleanliness.

This can be a closed-loop control system, if the level of cleanliness can be measured and compared with the desired cleanliness (reference input) and the difference is used to control the washing time of the machine.

1.2.3 Closed-Loop System versus Open-Loop System

An open-loop system can be modified as a closed-loop system by providing feedback. The provision of feedback automatically corrects the changes in output due to disturbances. Hence the closed-loop control system is called an automatic control system.

An advantage of the closed-loop control system is the fact that, the use of feedback makes the system response relatively insensitive to external disturbances and internal variations in system parameters. It is thus possible to use relatively inaccurate and inexpensive components to obtain the accurate control of a given plant, whereas doing so is impossible in the case of the open-loop system.

From the point of view of stability, the open-loop control system is easier to build because system stability is not a major problem. On the other hand, stability is a major problem in the closed-loop control system, which may tend to overcorrect errors that can cause oscillations of constant or changing amplitude.

It should be emphasized that for systems in which the inputs are known ahead of time and in which there are no disturbances, it is advisable to use open-loop control. Closed-loop control systems have advantages only when unpredictable disturbances and / or unpredictable variations in system components are present. Note that the output power rating partially determines the cost, weight, and size of a control system. The number of components used in a closed-loop control system is more than that for a corresponding open-loop control system. Thus, the closed-loop control system is generally higher in cost and power. To decrease the required power of a system, open-loop control may be used where applicable. A proper combination of open-loop and closed-loop controls is usually less expensive and will give satisfactory overall system performance.

The advantages and disadvantages of open-loop and closed-loop control systems are summarized in Table 1.1.

Table 1.1 Open-loop vs closed-loop

Open-loop control system	*Closed-loop control system*
1. The open-loop systems are simple and economical.	1. The closed-loop systems are complex and costlier.
2. They consume less power.	2. They consume more power.
3. The open-loop systems are easier to construct because of less number of components required.	3. The closed-loop systems are not easy to construct because of more number of components required.
4. Stability is not a major problem in open-loop control systems. Generally, the open-loop systems are stable.	4. Stability is a major problem in closed-loop control systems and more care is needed to design a stable closed-loop system.
5. The open-loop systems are inaccurate and unreliable.	5. The closed-loop systems are accurate and more reliable.
6. The changes in the output due to external disturbances are not corrected automatically. So they are more sensitive to noise and other disturbances.	6. The changes in the output due to external disturbances are corrected automatically. So they are less sensitive to noise and other disturbances.
	7. The feedback reduces the overall gain of the system.
	8. The feedback in a closed-loop system may lead to oscillatory response, because it may over correct errors, thus causing oscillations of constant or changing amplitude.

1.2.4 Linear versus Nonlinear Control Systems

The classification of control systems into linear and nonlinear is made according to the method of analysis and design. A system is said to be linear, if the principle of superposition applies. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is equal to the sum of the two individual responses. Hence for the linear system, the response to several inputs can be calculated by treating one input at a time and adding the results. Strictly speaking, linear systems do not exist in practice, since all physical systems are nonlinear to some extent. Linear feedback control systems are idealized models fabricated by the analyst purely for the simplicity of analysis and design. When the magnitudes of signals in a control system are limited to ranges in which system components exhibit linear characteristics, the system is essentially linear, but when the magnitudes of signals are extended beyond the range of the linear operation, depending on the severity of the nonlinearity, the system should no longer be considered linear. Common nonlinear effects focused in control systems are saturation, backlash, dead play between coupled gear members, nonlinear spring characteristics, nonlinear friction force or torque between moving members, and so on. Quite often, nonlinear characteristics are intentionally introduced in a control system to improve its performance, or to provide more effective control. For instance, to achieve minimum time control, an on-off type of controller is used in many missile or spacecraft control systems.

For linear systems, there exist a wealth of analytical and graphical techniques for design and analysis purposes. Nonlinear systems on the other hand are usually difficult to treat mathematically, and there are no general methods available for solving a wide class of nonlinear control systems.

1.2.5 Time-Invariant versus Time-Varying Control Systems

When the parameters of a control system are stationary with respect to time during the operation of the system, the system is called a *time-invariant system*. If the parameters of a control system vary with respect to time during the operation of the system, the system is called a *time-varying system*. In practice, most physical systems contain elements that drift or vary with time. For example, the winding resistance of an electric motor will vary when the motor is first being excited and its temperature is rising. Another example of a time-varying system is a guided missile control system in which the mass of the missile decreases as the fuel on board is being consumed during flight. The analysis and design of linear time-varying systems are usually much more complex than that of the linear time-invariant systems.

1.2.6 Continuous-Data versus Discrete-Data Control Systems

A continuous-data control system is one in which the signals at various parts of the system are all functions of the continuous time variable and among all continuous-data control systems, the signals may be further classified as ac or dc. An ac control system usually means that the signals in the system are modulated by some form of modulation scheme. On the other hand, when a dc control system is referred to, it does not mean that all the signals in the system are unidirectional, then there would be no corrective control movement. A dc control system simply implies that the signals are unmodulated, but they are still ac signals according to the conventional definition.

In practice, not all control systems are strictly of the ac or dc type. A system may incorporate a mixture of ac and dc components using modulators and demodulators to match the signals at various points in the system.

Discrete-data control systems differ from the continuous-data control systems in that the signals at one or more points of the system are in the form of either a pulse train or digital code. Usually discrete-data control systems are subdivided into sampled-data control systems and digital control systems. Sampled-data control systems refer to a more general class of discrete-data control systems, in which the signals are in the form of pulse data. A digital control system refers to the use of a digital computer or controller in the system so that the signals are digitally coded, such as in binary code. In general, a sampled-data control system receives data or information only intermittently at specific instants of time. Strictly, a sampled-data system can also be classified as an ac system, since the signal of the system is pulse modulated.

Sampling may be inherent or intentional. There are many advantages of incorporating sampling into a control system. One important advantage of the sampling operation is that expensive equipment used in the system may be time shared among several control channels. Another advantage is that pulse data are usually less susceptible to noise.

Because digital computers provide many advantages in size and flexibility, computer control has become increasingly popular in recent years.

1.3 WHAT FEEDBACK IS AND WHAT ITS EFFECTS ARE

One use of feedback is for the purpose of reducing the error between the reference input and the system output. The reduction of system error is merely one of the many important effects that feedback may have upon a system. Feedback also has effects on such system performance characteristics as stability, bandwidth, overall gain, disturbance and sensitivity.

When feedback is deliberately introduced for the purpose of control, its existence is easily identified. However, there are numerous situations wherein a physical system that we normally recognize as an inherently nonfeedback system turns out to have feedback when it is observed in a certain manner.

In general, we can state that whenever a closed sequence of cause-and-effect relationships exist among the variables of a system, feedback is said to exist. This view point will inevitably admit feedback in a large number of systems that ordinarily would be identified as nonfeedback systems.

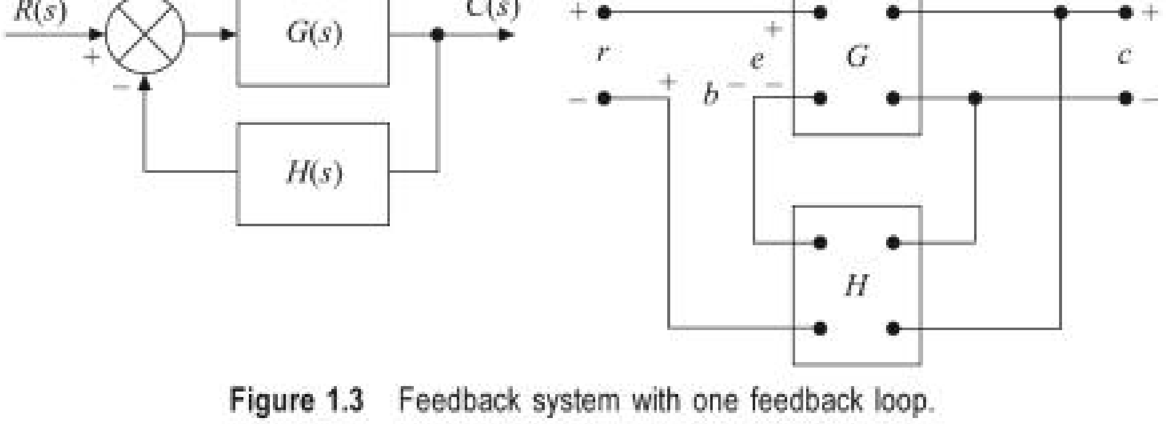


Figure 1.3 Feedback system with one feedback loop.

The input-output relation of a single loop control system shown in Figure 1.3 is given by

$$M = \frac{c}{r} = \frac{G}{1+GH} \tag{1.1}$$

1.3.1 Effect of Feedback on Overall Gain

As seen from Eq. (1.1), feedback affects the gain G of a nonfeedback system by a factor $1/(1 + GH)$. The system of Figure 1.3 is said to have negative feedback, since a minus sign is assigned to the feedback signal. The quantity GH may itself include a negative sign, so the general effect of feedback is that it may increase or decrease the gain. In a practical control system, G and H are functions of frequency, so the magnitude of $1 + GH$ may be greater than 1 in one frequency range but less than 1 in another. Therefore, feedback could increase the system gain in one frequency range but decrease it in another.

1.3.2 Effect of Feedback on Stability

Stability is a notion that describes whether the system will be able to follow the input command, or be used in general. In a nonrigorous manner, a system is said to be unstable if its output is out of control. To investigate the effect of feedback on stability, refer to Eq. (1.1). If $GH = -1$, the output of the system is infinite for any finite input and the system is said to be unstable. Therefore, we may state that feedback can cause a system that is originally stable to become unstable. Certainly, feedback is a two-edged sword, when used improperly, it can be harmful. In general, $GH = -1$ is not the only condition for unstability.

One of the advantages of incorporating feedback is that it can stabilize an unstable system. Let us assume that the feedback system shown in Figure 1.3 is unstable because $GH = -1$. If we introduce another feedback loop through a negative feedback gain of F as shown in Figure 1.4, the input-output relation of the overall system is

$$\frac{c}{r} = \frac{G}{1+GH+GF} \tag{1.2}$$

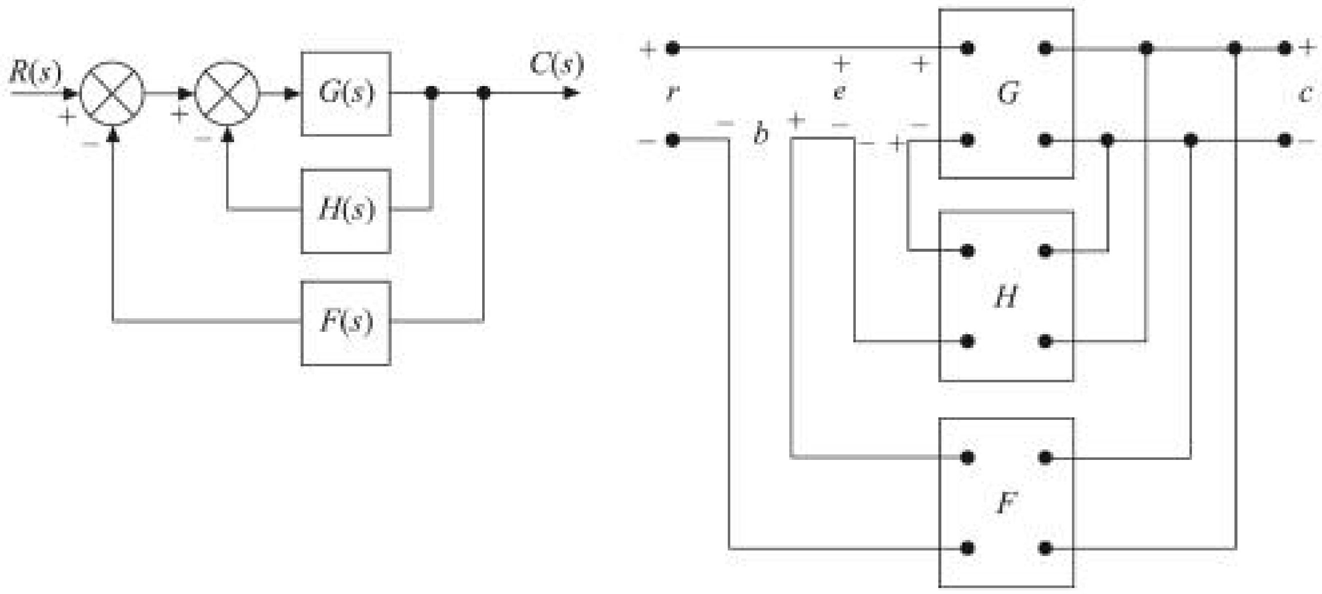


Figure 1.4 Feedback system with two feedback loops.

It is apparent that although the properties of G and H are such that the inner loop feedback system is unstable, because $GH = -1$, the overall system can be stable by proper selection of the outer loop feedback gain F. In practice, GH is a function of frequency, and the stability condition of the closed-loop system depends on the magnitude and phase of GH. So we can conclude that feedback can improve stability or be harmful to stability if it is not applied properly.

1.3.3 Effect of Feedback on External Disturbance or Noise

All physical systems are subject to some types of extraneous signals or noise during operation. Examples of these signals are thermal noise voltage in electronic circuits and brush or commutator noise in electric motors. External disturbances, for example wind gust acting on an antenna, are also quite common in control systems. Therefore, in the design of a control system, considerations should be given so that the system is insensitive to noise and disturbances and sensitive to input commands.

The effect of feedback on noise and disturbance depends greatly on where these extraneous signals occur in the system. No general conclusions can be reached, but in many situations, feedback can reduce the effect of noise and disturbance on system performance. Refer to Figure 1.5 in which r denotes the command signal and n is the noise signal. In the absence of feedback, $H = 0$, the output c due to n acting alone is

$$c = G_2 n \tag{1.3}$$

With the presence of feedback, the system output due to n acting alone is

$$c = \frac{G_2}{1 + G_1 G_2 H} n \tag{1.4}$$

Comparing Eqs. (1.3) and (1.4), we can conclude that the noise component in the output of Eq. (1.4) is reduced by a factor $1 + G_1G_2H$, if the latter is greater than unity and the system is kept stable.

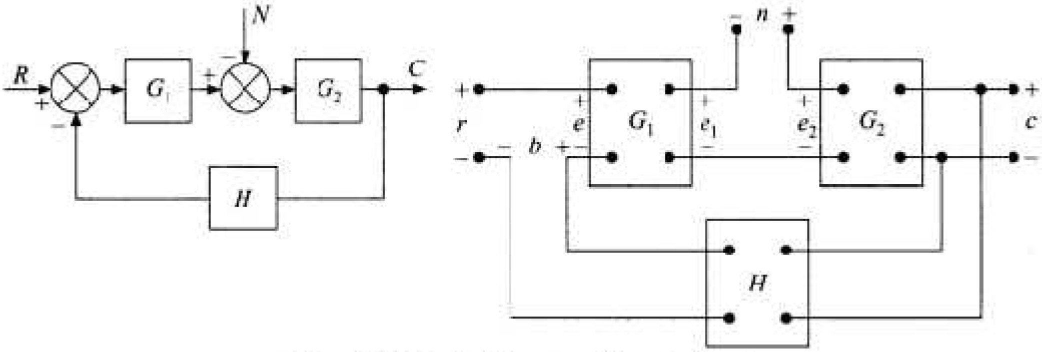


Figure 1.5 Feedback system with noise signal.

1.3.4 Effect of Feedback on Sensitivity

Sensitivity considerations often are important in the design of control systems. Since all physical elements have properties that change with environment and age, we cannot always consider the parameters of a control system to be completely stationary over the entire operating life of the system. For instance, the winding resistance of an electric motor changes as the temperature of the motor rises during operation.

In general, a good control system should be very insensitive to parameter variations but sensitive to input commands. Let us investigate what effect feedback has on the sensitivity to parameter variations. Referring to the general block diagram of a closed-loop control system of Figure 1.3, we consider G to be a gain parameter that may vary. The sensitivity of the gain of the overall system, M, to the variation in G is defined as

$$S_G^M = \frac{\partial M/M}{\partial G/G} = \frac{\text{Percentage change in } M}{\text{Percentage change in } G} \tag{1.5}$$

where ∂M denotes the incremental change in M due to the incremental change in G, ∂G.

$$S_G^M = \frac{\partial M}{\partial G} \cdot \frac{G}{M} = \frac{1}{1+GH} \tag{1.6}$$

This relation shows that if GH is a positive constant, the magnitude of the sensitivity function can be made arbitrarily small by increasing GH, provided that the system remains stable. It is apparent that in an open-loop system, the gain of the system will respond in a one-to-one fashion to the variation in G. (i.e. $S_G^M = 1$). In practice, GH is a function of frequency: The magnitude of $1 + GH$ may be less than unity over some frequency ranges, so that feedback could be harmful to the sensitivity to parameter variations in certain cases. In general, the sensitivity of the system gain of a feedback system to parameter variations depends on where the parameter is located.

1.4 SERVOMECHANISM

In modern usage, the term servomechanism or servo is restricted to feedback control systems in which the controlled variable is mechanical position or time derivatives of position, e.g. velocity and acceleration. Few servo mechanisms are illustrated below.

1.4.1 Automatic Tank Level Control System

Figure 1.6 shows an automatic tank level control system. The purpose of this system is to maintain the liquid level h (output) in the tank as close to the desired liquid level H as possible, even when the output flow rate is varied by opening the valve V_1. This has to be done by controlling the opening of the valve V_2. The potentiometer acts as an error detector. The slider arm A is positioned corresponding to the desired liquid level H (the reference input). The power amplifier and the motor drive form the control elements. The float forms the feedback path element. The valve V_2 to be controlled is the plant.

The liquid level is sensed by a float and it positions the slider arm B on the potentiometer. When the liquid level rises or falls, the potentiometer gives an error voltage proportional to the change in liquid level. The error voltage actuates the motor through a power amplifier which in turn conditions the plant (i.e. decreases or increases the opening of the valve V_2) in order to restore the desired liquid level. Thus, the control system automatically attempts to correct any deviation between the actual and desired liquid levels in the tank.

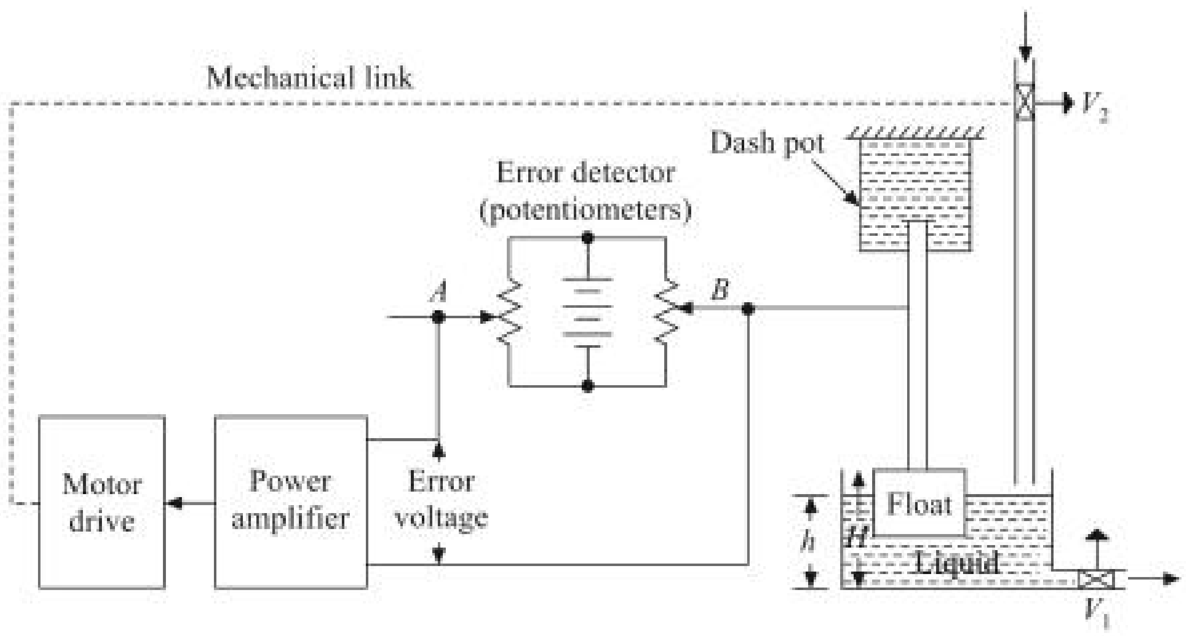


Figure 1.6 Automatic tank level control system.

1.4.2 A Position Control System

Figure 1.7 shows a servosystem used to position a load shaft. In this, the driving motor is geared to the load to be moved. The potentiometer is used as the error detector. The output and desired

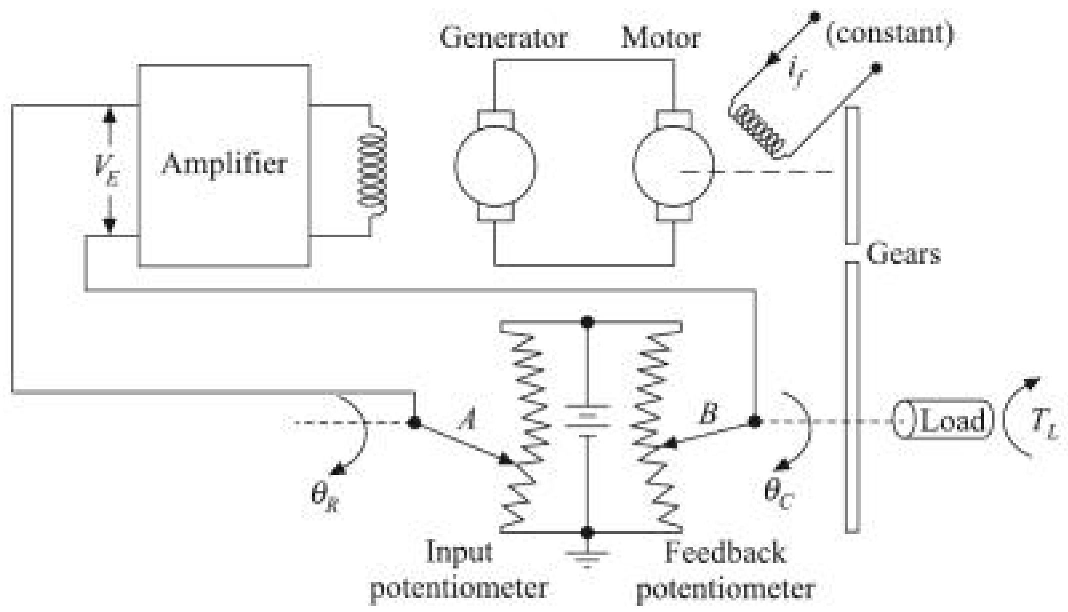


Figure 1.7 A position control system.

positions θ_C and θ_R respectively are measured and compared by the potentiometer pair whose output voltage V_E is proportional to the error in angular position $\theta_E = \theta_R - \theta_C$. The voltage $V_E = K_P\theta_E$ is amplified and is used to control the field current of a dc generator, which supplies armature voltage to the driving motor.

The position control systems have innumerable applications, namely machine tool position control, control of sheet metal thickness in hot rolling mills, radar tracking systems, missile guidance systems, etc.

1.4.3 DC Closed-Loop Control System

Figure 1.8 shows a typical dc (unmodulated) control system. The output signal θ_y represents the actual load position and the reference input θ_r represents the desired position of the load. A potentiometer error detector is used. The electrical error signal proportional to the difference in the positions of the actual and desired load positions is amplified by the dc amplifier and this output drives the dc motor which in turn through the gear box decides the position of the load. The signals are all unmodulated (i.e. dc).

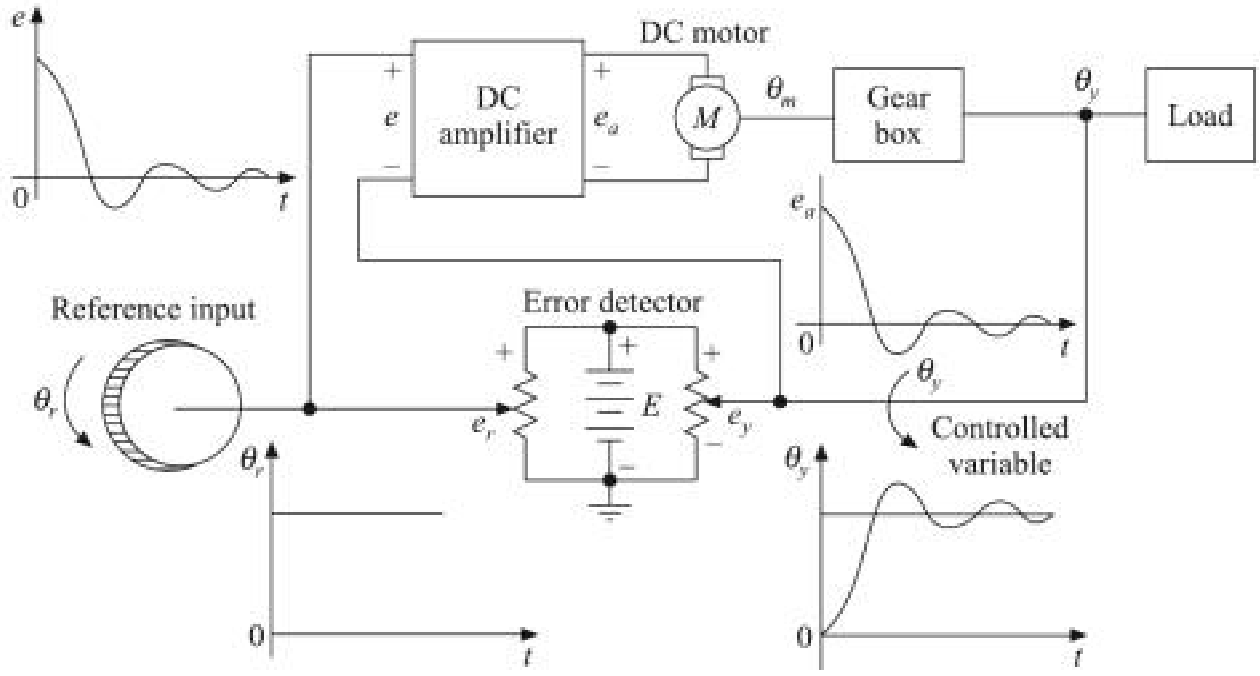


Figure 1.8 Schematic diagram of a typical dc closed-loop system.

1.4.4 AC Closed-Loop Control System

Figure 1.9 shows the schematic diagram of a typical ac control system. The output signal θ_y representing the load position is applied to the synchro control transformer. The reference input θ_r representing the desired output is applied to the synchro transmitter. The synchro pair acts as an error detector. The error signal is amplified by an ac amplifier and drives the ac servomotor which in turn positions the load through the gear box. The signals in this system are modulated (i.e. ac type).

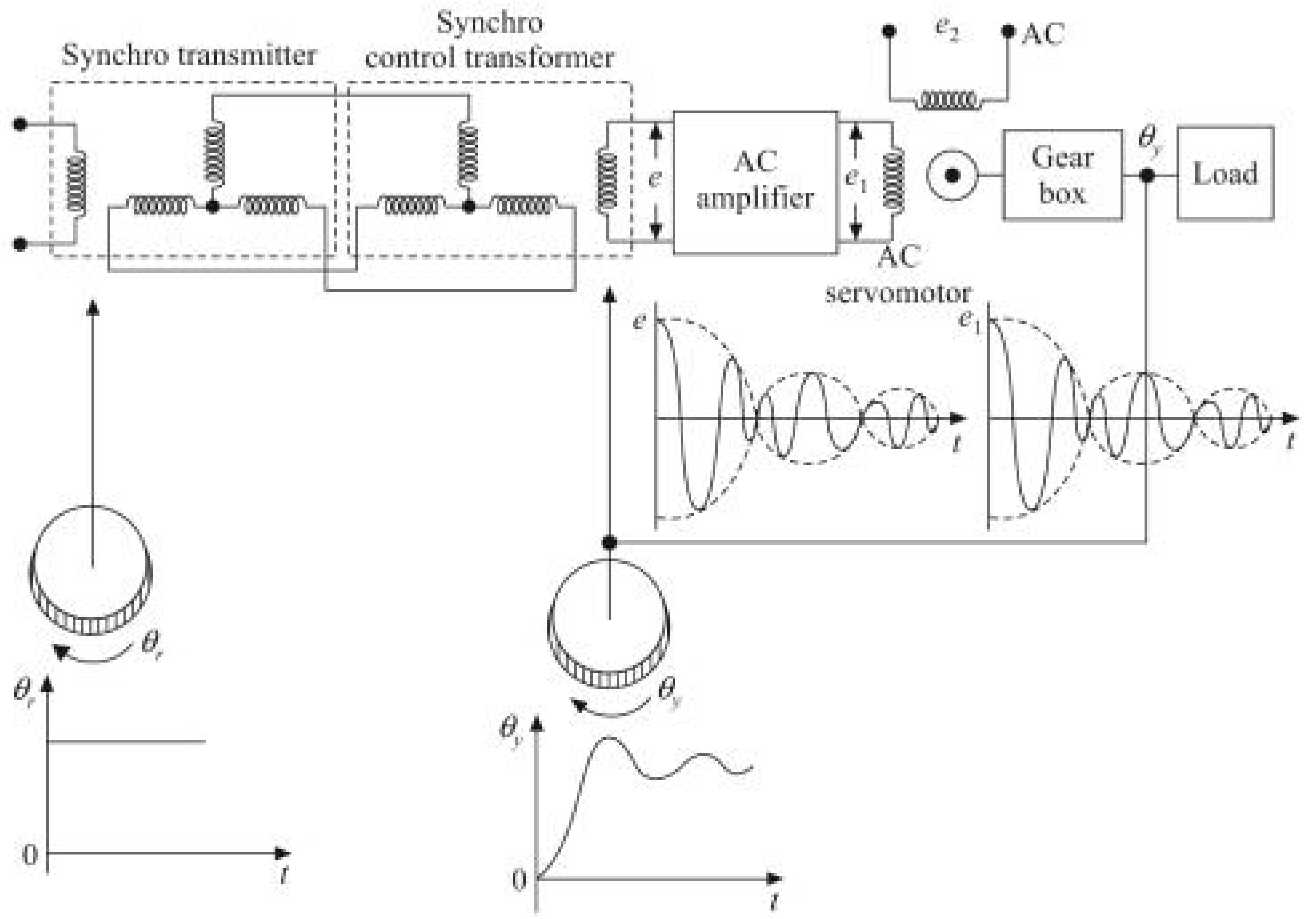


Figure 1.9 Schematic diagram of a typical ac closed-loop control system.

SHORT QUESTIONS AND ANSWERS

1. What do you mean by a system?

A. A system is a collection of objects (components) connected together to serve an objective, or a system is a combination of components that act together to perform a certain objective.

2. What do you mean by a control system?

A. A control system is that means by which any quantity of interest in a machine, mechanism, or some other equipment is maintained or altered in accordance with a desired manner. *or* A control system is a system in which the output quantity is controlled by varying the input quantity.

3. What do you mean by a physical system?

A. A physical system is a collection of physical objects connected together to serve an objective.

4. What do you mean by a physical model?

A. An idealized physical system is called a physical model.

5. What do you mean by a mathematical model?

A. The mathematical representation of the physical model is called the mathematical model.

6. When do you say that the mathematical model is linear?

A. A mathematical model is said to be linear, if the differential equation describing it has coefficients which are either functions only of the independent variable or are constants.

7. When do you say that the model is linear time-varying?

A. If the coefficients of the differential equation describing a system are functions of time (the independent variable), then the mathematical model is said to be linear time varying.

8. When do you say that the model is linear time-invariant?

A. If the coefficients of the differential equation describing a system are constants, then the model is said to be linear time-invariant.

9. How are control systems classified?

A. Control systems may be classified in a number of ways depending on the purpose of classification.

(a) Based on the hierarchy, control systems may be classified as
   (i) open-loop control systems
   (ii) closed-loop control systems
   (iii) optimal control systems
   (iv) adaptive control systems
   (v) learning control systems

(b) Based on the presence of human being as part of the system, control systems may be classified as
   (i) manually controlled systems
   (ii) automatic control systems

(c) Depending on the presence of feedback, control systems may be classified as
   (i) open-loop control systems
   (ii) closed-loop control systems or feedback control systems

(d) Based on the main purpose of the system, control systems may be classified as
   (i) position control systems
   (ii) velocity control systems
   (iii) traffic control systems
   (iv) temperature control systems, etc.

10. How are feedback control systems classified?

A. Feedback control systems may be classified in a number of ways depending on the purpose of classification.

(a) According to the method of analysis and design, control systems may be classified as linear control systems and nonlinear control systems.

(b) Depending on whether the parameters of the system remain constant or vary with time, control systems may be classified as time-invariant control systems and time-varying control systems.

(c) According to the types of signals used in the system, control systems may be classified as continuous-data control systems and discrete-data control systems or ac (modulated) control systems and dc (unmodulated) control systems.

(d) Depending on the application, control systems may be classified as position control systems, velocity control systems, etc.

(e) Depending on the number of inputs and outputs, control systems may be classified as single-input-single-output (SISO) control systems and multi-input-multi-output (MIMO) control systems. MIMO systems are also called multivariable systems.

(f) Depending on the number of open-loop poles of the system transfer function present at the origin of the s-plane, control systems may by classified as

   (i) Type-0
   (ii) Type-1,
   (iii) Type-2, etc. control systems.

(g) Depending on the order of the differential equation used to describe the system, control systems may be classified as first-order control systems, second-order control systems, etc.

(h) Depending on the type of damping, control systems may be classified as

   (i) undamped systems,
   (ii) underdamped systems,
   (iii) criticallydamped systems, and
   (iv) overdamped systems.

11. What are the two major types of control systems?

A. The two major types of control systems are (a) open-loop control systems and (b) closed-loop control systems.

12. What do you mean by an open-loop control system?

A. An open-loop control system is one in which the output quantity has no effect on the input quantity, that means, the output is not fed back to the input for correction, i.e. a system in which there is no feedback is called an open-loop control system.

13. What do you mean by a closed-loop control system?

A. A closed-loop control system is one in which the output has an effect on the input. The output is fed back, compared with the reference input and the difference between them is used to control and bring the output of the system to a desired level.

14. What do you mean by feedback?

A. Feedback is a control action in which the output is sampled and a proportional signal is given to input for automatic correction of any changes in system output.

15. Which feedback is employed in control systems?

A. Negative feedback is employed in control systems.

16. What are the basic components of a closed-loop control system?

A. The basic components of a feedback control system are: plant, feedback path elements, error detector and controller.

17. Why negative feedback is preferred in control systems?

A. Negative feedback is invariably preferred in closed-loop control systems because negative feedback results in better stability in steady-state and rejects any disturbance signals. It also has low sensitivity to parameter variations.

18. What are the characteristics of negative feedback?

A. The characteristics of negative feedback are as follows:

(a) Accuracy in tracking steady-state value
(b) Rejection of disturbance signals
(c) Low sensitivity to parameter variations
(d) Reduction in gain at the expense of better stability

19. What is the effect of positive feedback on stability?

A. The effect of positive feedback on stability is—positive feedback increases the error signal and drives the output to instability, but some times positive feedback is used in minor loops in control systems to amplify certain internal signals or parameters.

20. Compare open-loop and closed-loop control systems.

A. Comparison of open-loop and closed-loop control systems is given in the following table:

Open-loop control systems	*Closed-loop control systems*
1. Inaccurate and unreliable.	1. Accurate and reliable.
2. Consume less power.	2. Consume more power.
3. Simple and economical.	3. Complex and costlier.
4. The changes in output due to external disturbances are not corrected automatically.	4. The changes in output due to external disturbances are corrected automatically.
5. They are generally stable.	5. Efforts are needed to design a stable system.

21. Distinguish between linear and nonlinear control systems.

A. A linear control system is one for which the principle of superposition and the principle of homogeneity are valid and a nonlinear control system is one for which the principle of superposition and the principle of homogeneity are not valid.

22. State the principle of superposition.

A. The principle of superposition states that the response produced by the simultaneous application of two different forcing functions is equal to the sum of the two individual responses.

23. Distinguish between time-invariant and time-varying control systems.

A. A time-invariant control system is one in which the parameters of the system are stationary with respect to time during the operation of the system. Its output characteristics do not change with time and it can be represented by constant coefficient differential equations.

A time-varying control system is one in which the parameters of the system are not stationary with respect to time during the operation of the system, i.e. the parameters of the system vary with time. Its output characteristics change with time and the coefficients of its differential equation are functions of time.

24. Distinguish between continuous-data and discrete-data control systems.

A. A continuous-data control system is one in which the signals at various parts of the system are all functions of the continuous time variable.

A discrete-data control system is one in which the signals at one or more points in the system are either in the form of a pulse train or a digital code.

25. Distinguish between ac and dc control systems.

A. The ac control systems are those in which the signals are modulated. The dc control systems are those in which the signals are not pure dc, but they are unmodulated.

26. Distinguish between sampled-data and digital control systems.

A. A sampled-data control system refers to a more general class of discrete-data control systems in which the signals are in the form of pulsed-data.

A digital control system is one which uses a digital computer or controller so that the signals are digitally coded.

27. What is the advantage of sampling?

A. One important advantage of the sampling operation is that expensive equipment used in the system may be time shared among several control channels.

28. What is the effect of feedback on overall gain?

A. The feedback affects the gain G of a nonfeedback system by a factor $1/(1 + GH)$. In a practical control system, G and H are functions of frequency, so the magnitude of $1 + GH$ may be greater than 1 in one frequency range but less than 1 in another. Therefore, feedback could increase the system gain in one frequency range but decrease it in another.

29. What is the effect of feedback on stability?

A. Feedback may cause a system that is originally stable to become unstable. In some cases feedback can stabilise an unstable system. So feedback is a two-edged sword, when used improperly it can be harmful.

30. What is the effect of feedback on external disturbance?

A. The effect of feedback on noise and disturbance depends greatly on where these extraneous signals occur in the system. No general conclusions can be reached, but in many situations feedback can reduce the effect of noise and disturbance on system performance.

31. What is the effect of feedback on sensitivity?

A. In general, a good control system should be very insensitive to parameter variations but sensitive to input commands. Feedback could be harmful to the sensitivity to parameter variations in certain cases. In general, the sensitivity of the system gain of a feedback system to parameter variations depends on where the parameter is located.

REVIEW QUESTIONS

1. With a neat diagram explain the working of an automatic tank level control system.
2. With a neat diagram explain the working of a position control system.
3. With a neat diagram explain the working of a dc position control system.
4. With a neat diagram explain the working of an ac position control system.

FILL IN THE BLANKS

1. A physical system is a collection of __________ connected together.
2. An idealized physical system is called a _________.
3. The mathematical representation of the __________ is called the mathematical model.
4. Based on the hierarchy, control systems may be classified as (i) __________ (ii) _________ (iii) __________ (iv) __________ and (v) ____________.
5. Based on the presence of human being as part of the system, control systems may be classified as (i) ___________ and (ii) ___________.
6. Depending on the presence of feedback, control systems may be classified as (i) ___________ and (ii) ___________.
7. Based on the main purpose of the system, control systems may be classified as (i) ___________ (ii) ___________ (iii) ____________, etc.
8. According to the method of analysis and design, control systems may be classified as (i) ___________ and (ii) ___________.
9. Depending on whether the parameters of the system remain constant or vary with time, control systems may be classified as (i) ___________ and (ii) ___________.
10. According to the type of signals used in the system, control systems may be classified as (i) ___________ and (ii) ___________ or (i) ___________ and (ii) ___________.
11. Depending on the application, control systems may be classified as (i) ___________ (ii) ___________ (iii) ___________, etc.
12. Depending on the number of inputs and outputs, control systems may be classified as (i) ___________ and (ii) ___________.
13. Depending on the number of open-loop poles of the system transfer function present at the origin of the s-plane, control systems may be classified as (i) ________ (ii) _______ (iii) _______ systems, etc.
14. Depending on the order of the differential equation used to describe the system, control systems may be classified as (i) _________ (ii) _________ (iii) _________ etc.
15. Depending on the type of damping, control systems may be classified as (i) _______ (ii) _________ (iii) _________ and (iv) _________.
16. The two major types of control systems are (i) _________ and (ii) _________.

17. An open-loop control system is one in which the _________ has no effect on the ________.

18. A closed-loop control system is one in which the ________ has an effect on the ________.

19. ________ feedback is employed in control systems.

20. The basic components of a feedback control system are (i) _______ (ii) _______ (iii) ________ and (iv) _________.

21. A linear control system is one for which the _________ and the __________ are valid.

22. In ac systems, the signals are _________ whereas in dc systems the signals are _____.

23. Sampling may be _________ or _________.

Answers to Fill in the Blanks

1. physical objects **2.** physical model **3.** physical model **4.** (i) open-loop control systems, (ii) closed-loop control systems, (iii) optimal control systems, (iv) adaptive control systems, (v) learning control systems **5.** (i) manually controlled systems, (ii) automatic control systems **6.** (i) open-loop control systems, (ii) closed-loop control systems **7.** (i) position control systems, (ii) velocity control systems, (iii) traffic control systems, etc. **8.** (i) linear control systems, (ii) nonlinear control systems **9.** (i) time-varying control systems, (ii) time-invariant control systems **10.** (i) continuous-data control systems, (ii) discrete-data control systems, or (i) ac control systems, (ii) dc control systems **11.** (i) position control systems, (ii) velocity control systems, (iii) traffic control systems, etc. **12.** (i) single-input-single-output control systems, (ii) multi-input-multi-output control systems **13.** (i) type-0, (ii) type-1, (iii) type-2 **14.** (i) first-order control systems, (ii) second-order control systems, (iii) third-order control systems, etc. **15.** (i) undamped systems, (ii) underdamped systems, (iii) critically-damped systems, (iv) overdamped systems **16.** (i) open-loop control systems, (ii) closed-loop control systems **17.** output quantity, input quantity **18.** output, input **19.** negative **20.** (i) plant, (ii) feedback path elements, (iii) error detector, (iv) controller **21.** principle of superposition, principle of homogeneity **22.** modulated, unmodulated **23.** intentional, inherent.

2

Mathematical Models of Physical Systems

2.1 MODELLING OF MECHANICAL SYSTEM ELEMENTS

Most control systems contain mechanical as well as electrical components, although some systems even have hydraulic and pneumatic elements. From a mathematical view point, the descriptions of mechanical and electrical elements are analogous. In fact, we can show that given an electrical device, there is usually an analogous mechanical counterpart mathematically and vice versa.

The motion of mechanical elements can be described in various dimensions as translational, rotational, or combinations. The equations governing the motion of mechanical systems are often formulated directly or indirectly from Newton's law of motion. Mechanical translational systems are those in which the motion takes place along a straight line. The variables that are used to describe the translational motion are acceleration, velocity and displacement. Newton's law of motion for translational systems states that the algebraic sum of forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction. This law can be expressed as

$$\sum \text{Forces} = Ma(t) = M\frac{dv(t)}{dt} = M\frac{d^2x(t)}{dt^2}$$

where M denotes the mass of the body and $a(t)$ is the acceleration of the body in the direction considered, $v(t)$ is the linear velocity of the body and $x(t)$ is the displacement of the body.

Mechanical translational systems are modelled by three ideal elements: mass, spring, and damper.

1. Mass: Mass is considered as a property of an element that stores the kinetic energy of translation motion. If W denotes the weight of a body, the mass M is given by

$$M = \frac{W}{g}$$

where g is the acceleration of free fall of the body due to gravity. One end of mass is always connected to the ground.

2. Linear Spring: In practice, a linear spring may be a model of a natural spring or a compliance of a cable or belt. In general, a spring is considered to be an element that stores potential energy. All springs in real life are nonlinear to some extent. However, if the deformation of the spring is small, its behaviour can be approximated by a linear relationship.

3. Friction: Whenever there is motion or tendency of motion between two physical elements, frictional forces exist. The frictional forces encountered in physical systems are usually of a nonlinear nature. Three different types of friction are commonly used in practical systems: viscous friction, static friction and coulomb friction.

(a) *Coulomb friction force:* This is the force of sliding friction between dry surfaces. Coulomb friction force is substantially constant.

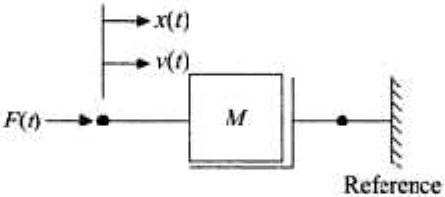
(b) *Viscous friction force:* This is the force of friction between moving surfaces separated by viscous fluid or the force between a solid body and a fluid medium. Viscous friction force is approximately linearly proportional to velocity over a certain limited velocity range.

(c) *Stiction:* This is the force required to initiate motion between two contacting surfaces.

In most physical situations of interest, the viscous friction predominates. The friction force acts in a direction opposite to that of velocity. The element for viscous friction is often represented by a dash pot.

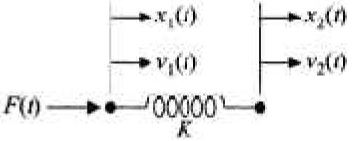
The translational elements and the corresponding equations of motion are shown in Figure 2.1(a).

1. The mass element



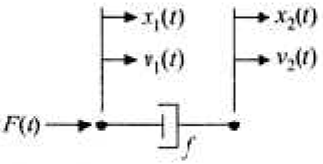
$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$

2. The spring element



$$F = K(x_1 - x_2) = Kx$$
$$= K\int_{-\infty}^{t}(v_1 - v_2)\,dt = K\int_{-\infty}^{t} v\,dt$$

3. The damper element



$$F = f(v_1 - v_2) = fv$$
$$= f(\dot{x}_1 - \dot{x}_2) = f\dot{x}$$

The units are x(m), v(m/s), M(kg), F(N), K(N/m), f(N/m/s)

Figure 2.1(a) Ideal elements for mechanical translational systems.

Mechanical rotational systems are those in which the motion is about a fixed axis. Newton's law of motion for rotational systems states that the algebraic sum of moments or torques about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis; or

$$\sum \text{Torques} = J\alpha(t) = J\frac{d\omega(t)}{dt} = J\frac{d^2\theta(t)}{dt^2}$$

where $\theta(t)$ is the angular displacement, $\omega(t)$ is the angular velocity and $\alpha(t)$ is the angular acceleration.

Mechanical rotational systems are modelled by three ideal elements: inertia, torsional spring, and damper.

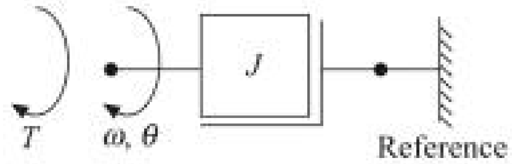
1. Inertia: Inertia, J, is considered to be the property of an element that stores the kinetic energy of rotational motion. The inertia of a given element depends on the geometric composition about the axis of rotation and its density. One end of the inertia element is always connected to ground.

2. Torsional spring: As with the linear spring for translational motion, a torsional spring constant K can be deviced to represent the compliance of a rod or a shaft when it is subjected to an applied Torque.

3. Friction: The three types of friction for rotational systems are (a) coulomb friction, (b) viscous friction, and (c) stiction—same as for translational systems. Out of these three, viscous friction is the dominant one. This element is often represented by a dash pot.

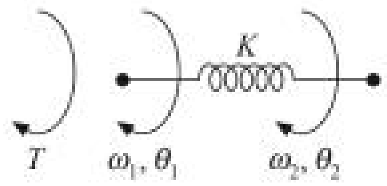
The rotational elements and the corresponding equations of motion are shown in Figure 2.1(b).

1. The inertia element



$$T = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$

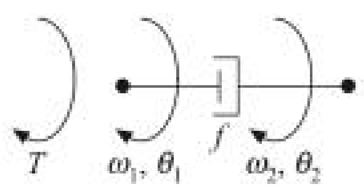
2. The torsional spring element



$$T = K(\theta_1 - \theta_2) = K\theta$$

$$= K\int_{-\infty}^{t}(\omega_1 - \omega_2)\,dt = K\int_{-\infty}^{t}\omega\,dt$$

3. The damper element



$$T = f(\omega_1 - \omega_2) = f\omega$$

$$= f(\dot{\theta}_1 - \dot{\theta}_2) = f\dot{\theta}$$

The units are θ (rad), ω (rad/s), J (kg-m^2), T (N-m), K (N-m/rad), f (N-m/rad/s)

Figure 2.1(b) Ideal elements for mechanical rotational systems.

2.1.1 Translational Systems

Consider the mechanical system shown in Figure 2.2 (a). It is simply a mass M attached to a spring (stiffness K) and a dash pot (viscous friction coefficient f) on which the force F acts. Displacement x is positive in the direction shown. Let v be the velocity. The zero position is taken to be at the point where the spring and mass are in static equilibrium.

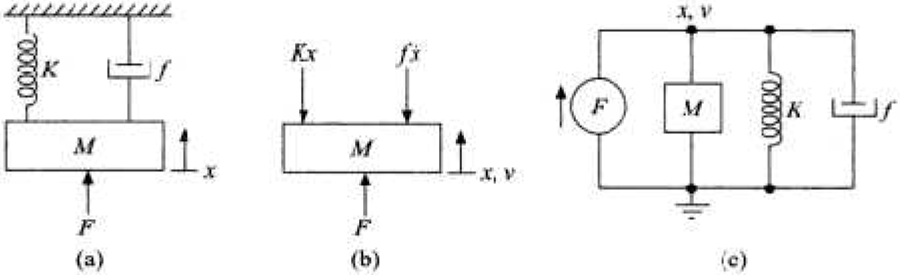


Figure 2.2 (a) Translational system, (b) free-body diagram, and (c) mechanical network.

The systematic way of analyzing such a system is to draw a free-body diagram or mechanical network as shown in Figures 2.2(b) and 2.2(c) respectively. Then, by applying Newton's law of motion (which states that the algebraic sum of the forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction) to the free-body diagram or mechanical network, the force equation can be written in terms of displacement x or velocity v as follows:

$$F - f\frac{dx}{dt} - Kx = M\frac{d^2x}{dt^2} \quad \text{or} \quad F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx$$

i.e.

$$F - fv - K\int v\,dt = M\frac{dv}{dt} \quad \text{or} \quad F = M\frac{dv}{dt} + fv + K\int v\,dt$$

This is a linear, constant coefficient differential equation describing the dynamics of the mechanical system shown in Figure 2.2(a).

The free-body diagram is drawn like this: the free-body diagram for a translational system indicates the elements masses M and all the forces acting on them. For the system shown in Figure 2.2(a), there is only one mass M. An external force F is acting on it to move it by a distance x or with a velocity v. This motion is opposed by the force components of mass M, i.e. $M\ddot{x}$ and the force components due to spring, i.e. Kx and damper, i.e. $f\dot{x}$. So, the free-body diagram is as shown in Figure 2.2(b).

In the free-body diagram, F is the external force applied on the mass M. It is being opposed by the forces $f\dot{x}$ and Kx. The algebraic sum of these three forces, i.e. $F - Kx - f\dot{x}$ is equal to the product of the mass and acceleration of the body, i.e. $M\ddot{x}$. Therefore,

$$F - Kx - f\dot{x} = M\ddot{x}$$

is the governing equation.

The mechanical network for the given translational system of Figure 2.2(a) is drawn like this: one end of F is to be grounded. Also one end of M is to be grounded. So one end of F and M is grounded. The other end of F is connected to the other end of M because F is applied on M. In the mechanical system, one end of K and f is grounded and the other ends of K and f are connected to the free end of M. So in the mechanical network also one end of K and f is grounded and the other ends are connected to the free end of M.

Once a mechanical network is drawn, the differential equations can be written very easily at each node assuming that the displacement or velocity at the node under consideration is higher than the displacement or velocity at all other nodes. In Figure 2.2, only one node is there, the other node is the reference or ground node. Only external force F is towards the node. All other forces, i.e. $M\ddot{x}$, $f\dot{x}$ and Kx are directed away from the node. Now the differential equation is like KCL equation for electrical circuits which says that the sum of currents coming into the node is equal to the sum of currents going out of the node. The force components are similar to currents. Therefore the force $F(t)$ directed towards the node is equal to the sum of the force components $M\ddot{x}$, $f\dot{x}$ and Kx directed away from the node. Therefore, the equation of motion is

$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx$$

2.1.2 Rotational Systems

Consider the mechanical rotational system shown in Figure 2.3(a) which consists of a rotatable disc of moment of inertia J and a shaft of stiffness K. The disc rotates in a viscous medium with viscous friction coefficient f.

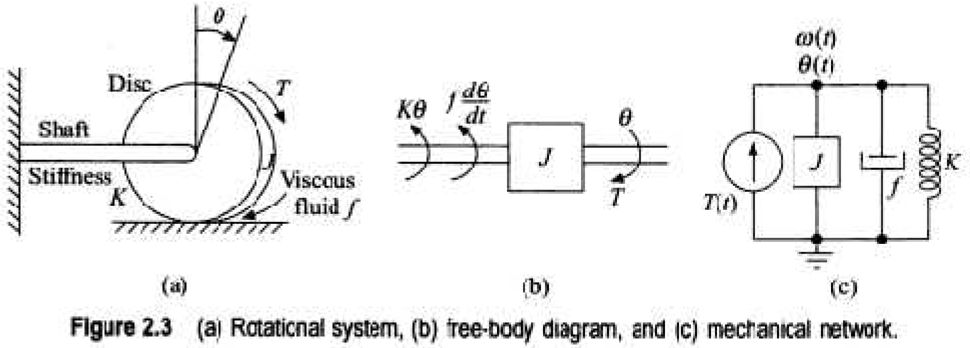


Figure 2.3 (a) Rotational system, (b) free-body diagram, and (c) mechanical network.

Let T be the applied torque which tends to rotate the disc. The free-body diagram and the mechanical network are shown in Figure 2.3(b) and Figure 2.3(c) respectively.

The free-body diagram is drawn like this: the free-body diagram for a rotational system indicates the inertia elements J and all the torques acting on them. For the system shown in

Figure 2.3(a), there is only one inertia element J. An external torque T is acting on it to move it by an angle θ or with an angular velocity ω. This angular motion is opposed by the torque component of J, i.e. $J\ddot{\theta}$ and the torque components due to damper, i.e. $f\dot{\theta}$ and spring, i.e. $K\theta$. So the free-body diagram is drawn as shown in Figure 2.3(b).

In the free-body diagram, T is the external torque applied on the inertia J. It is being opposed by the torques $f\dot{\theta}$ and $K\theta$. The algebraic sum of these three torques is $T - f\dot{\theta} - K\theta$ which is equal to the product of the inertia and angular acceleration of the body, i.e. $J\ddot{\theta}$. Therefore,

$$T - f\frac{d\theta}{dt} - K\theta = J\frac{d^2\theta}{dt^2}$$

is the governing equation.

The mechanical network for the given rotational system of Figure 2.3(a) is drawn like this: one end of T is to be grounded. Also one end of J is to be connected to ground. So one end of T and J is grounded. The other end of T is connected to the other end of J because T is applied on J. In the mechanical system of Figure 2.3(a), one end of f and K is grounded and the other ends are connected to the free end of J. So in the mechanical network also one end of f and K is grounded and the other end is connected to the free end of J.

Once a mechanical network is drawn, the differential equations can be written very easily at each node assuming that the angular displacement or angular velocity at the node under consideration is higher than the angular displacement or angular velocity at all other nodes. In Figure 2.3 only one node is there, the other node is the reference or ground node. Only external torque T is towards the node. All other torques, i.e. $J\ddot{\theta}$, $f\dot{\theta}$ and $K\theta$ are directed away from the node. Now the differential equation is like KCL equation for electrical circuits which says that the sum of currents coming into the node is equal to the sum of currents going out of the node. The torque components are similar to currents. Therefore, the torque T directed towards the node is equal to the sum of the torque components $J\ddot{\theta}$, $f\dot{\theta}$ and $K\theta$ directed away from the node. Therefore, the equation of motion is

$$T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} - K\theta$$

2.2 ELECTRICAL SYSTEMS

The resistor, inductor, and capacitor are the three basic elements of electrical circuits. These circuits are analyzed by the application of Kirchoff's voltage and current laws. Out of these three basic elements, inductor and capacitor are the energy storage elements and resistor is the energy dissipative element.

Consider the *R-L-C* series circuit shown in Figure 2.4(a). The equations describing the behaviour of the system in terms of current are as follows:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt = e$$

$$e_o = \frac{1}{C}\int i dt$$

$$e_o = e_c$$

In terms of electric charge, $q = \int i dt$, the describing equations are as follows:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = e$$

$$e_o = \frac{q}{c}$$

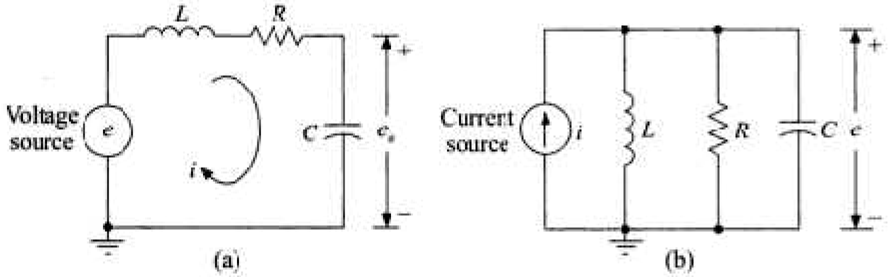


Figure 2.4 (a) *R-L-C* series circuit and (b) *R-L-C* parallel circuit

Similarly, for the *R-L-C* parallel circuit shown in Figure 2.4(b), the describing equation in terms of voltage using Kirchoff's current law is as follows:

$$C\frac{de}{dt} + \frac{e}{R} + \frac{1}{L}\int_{-\infty}^{t} e dt = i$$

In terms of magnetic flux linkage, $\phi = \int e\, dt$, the describing equation is as follows:

$$C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi = i$$

2.3 ANALOGOUS SYSTEMS

Comparing equations for the mechanical translational system or for the mechanical rotational system and for the series electrical system, it is seen that they are of identical form. Such systems whose differential equations are of identical form are called *analogous systems*. The force F (torque T) and voltage e are the analogous variables here. This is called the force (torque)-voltage analogy. A list of analogous variables in this analogy is given in Table 2.1.

Table 2.1 Analogous quantities in force (torque)-voltage analogy

Mechanical translational system	*Mechanical rotational system*	*Electrical system*
Force F	Torque T	Voltage e
Mass M	Moment of inertia J	Inductance L
Viscous friction coefficient f	Viscous friction coefficient f	Resistance R
Spring stiffness K	Torsional spring stiffness K	Reciprocal of capacitance $1/C$
Displacement x	Angular displacement θ	Charge q
Velocity v	Angular velocity ω	Current i

Similarly, comparing equations for the mechanical translational system or for the mechanical rotational system and for the parallel electrical system, it is seen that they are of identical form. In this case, force F (torque T) and current i are the analogous variables. This is called the force (torque)-current analogy. A list of analogous quantities in this analogy is given in Table 2.2.

Table 2.2 Analogous quantities in force (torque)-current analogy

Mechanical translational system	*Mechanical rotational system*	*Electrical system*
Force F	Torque T	Current i
Mass M	Moment of inertia J	Capacitance C
Viscous friction coefficient f	Viscous friction coefficient f	Reciprocal of resistance $1/R$
Spring stiffness K	Torsional spring stiffness K	Reciprocal of inductance $1/L$
Displacement x	Angular displacement θ	Magnetic flux linkage ϕ
Velocity v	Angular velocity ω	Voltage e

The concept of analogous system is a useful technique for the study of various systems like electrical, mechanical, thermal, hydraulic, etc. If the solution of one system is obtained, it can be extended to all other systems analogous to it. Generally, it is convenient to study a non-electrical system in terms of its electrical analog as electrical systems are more easily amenable to experimental study.

In this book, we study the systems using linear systems theory. There are two ways of justifying the linear systems approach. One is that the system is basically linear, or the system is operated in the linear region so that most of the conditions of linearity are satisfied. The second is that the system is basically nonlinear or operated in a nonlinear region, but to apply the linear analysis and design tools, we linearize the system about a nominal operating point. The analysis is applicable only for the range of variables in which the linearization is valid.

2.3.1 Impulse Response and Transfer Functions of Linear Systems

The classical way of modelling linear systems is to use transfer functions to represent input-output relations between variables. One way to define the transfer function is to use the impulse response which is defined as follows:

Impulse response: Consider that a linear time-invariant system has the input $r(t)$ and the output $c(t)$. The system can be characterized by its impulse response $g(t)$, which is defined as the output when the input is a unit impulse function $\delta(t)$. Once the impulse response of a linear system is known, the output of the system, $c(t)$, with any input $r(t)$ can be found by using the transfer function.

2.4 TRANSFER FUNCTION: SINGLE-INPUT-SINGLE-OUTPUT SYSTEMS

The transfer function of a linear time-invariant system is defined as the Laplace transform of the impulse response, with all the initial conditions set to zero. Let $G(s)$ denote the transfer function of a single-input-single-output system, with input $r(t)$ and output $c(t)$ and impulse response $g(t)$. Then the transfer function $G(s)$ is defined as

$$G(s) = L\ [g(t)]$$

The transfer function $G(s)$ is related to the Laplace transform of the input and the output through the following relation

$$G(s) = \frac{C(s)}{R(s)}$$

with all the initial conditions set to zero, and $C(s)$ and $R(s)$ are the Laplace transforms of $c(t)$ and $r(t)$ respectively. That is the transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input with all initial conditions neglected.

Although the transfer function of a linear system is defined in terms of the impulse response, in practice, the input-output relation of a linear time-invariant system with continuous data input is often described by a differential equation, so that it is more convenient to derive the transfer function directly from the differential equation. Let us consider that the input-output relation of a linear time-invariant system is described by the following nth order differential equation with constant real coefficients.

$$a_0 \frac{d^n c(t)}{dt^n} + a_1 \frac{d^{n-1} c(t)}{dt^{n-1}} + a_2 \frac{d^{n-2} c(t)}{dt^{n-2}} + \cdots + a_{n-1} \frac{dc(t)}{dt} + a_n c(t)$$

$$= b_0 \frac{d^m r(t)}{dt^m} + b_1 \frac{d^{m-1} r(t)}{dt^{m-1}} + \cdots + b_{m-1} \frac{dr(t)}{dt} + b_m r(t) \qquad (2.1)$$

Once the input $r(t)$ for $t \geq t_0$ and the initial conditions of $c(t)$ and derivatives of $c(t)$ are specified at the initial time $t = t_0$, the output response $c(t)$ for $t \geq t_0$ is determined by solving Eq. (2.1). However, the solution of higher-order differential equations is quite tedious. So the analysis and design of linear systems is done using transfer functions.

To obtain the transfer function of the linear system, simply take the Laplace transform on both sides of Eq. (2.1) and assume zero initial conditions.

The result is

$$(a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n)\, C(s) = (b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m)\, R(s)$$

The transfer function between $r(t)$ and $c(t)$ is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

The properties of the transfer function are as follows:

1. The transfer function is defined only for a linear time-invariant system. It is not defined for nonlinear systems.
2. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the impulse response.

   Alternatively, the transfer function between a pair of input and output variables of a system is the ratio of the Laplace transform of the output to the Laplace transform of the input.
3. All initial conditions of the system are set to zero.
4. The transfer function is independent of the input of the system.
5. The transfer function of a continuous-data system is expressed only as a function of the complex variable s. It is not a function of the real variable time, or any other variable that is used as the independent variable. For discrete-data systems modelled by difference equations, the transfer function is a function of z when the z-transform is used.

2.4.1 Proper Transfer Function

The transfer function is said to be strictly proper, if the order of the denominator polynomial is greater than that of the numerator polynomial (i.e. $n > m$). The transfer function is said to be proper, if the order of the numerator polynomial is equal to that of the denominator polynomial (i.e. $m = n$). The transfer function is said to be improper, if the order of the numerator polynomial is greater than that of the denominator polynomial (i.e. $m > n$).

2.4.2 Characteristic Equation

The characteristic equation of a linear system is defined as the equation obtained by setting the denominator polynomial of the transfer function to zero. Thus, the characteristic equation of the system described above is

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \cdots + a_{n-1}s + a_n = 0 \qquad (2.2)$$

Equation (2.2) is called the *characteristic equation* because it characterizes the behaviour of the system.

The stability of linear single-input-single-output systems is governed completely by the roots of the characteristic equation.

2.5 TRANSFER FUNCTION (MULTIVARIABLE SYSTEMS)

The definition of a transfer function is easily extended to a system with multiple inputs and outputs. A system of this type is often referred to as a *multivariable system*. In a multivariable system, a differential equation of the form described above may be used to describe the relationship between an input-output pair, when all other inputs are set to zero. Since the principle of superposition is valid for linear systems, the total effect on any output due to all the inputs acting simultaneously is obtained by adding up the outputs due to each input acting alone.

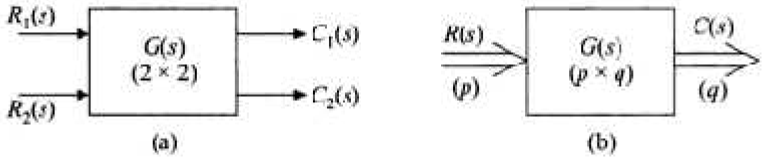


Figure 2.5 Multivariable systems.

If a system has two inputs and two outputs as shown in Figure 2.5(a), then the outputs are given by

$$C_1(s) = G_{11}(s)\, R_1(s) + G_{12}(s)\, R_2(s)$$

$$C_2(s) = G_{21}(s)\, R_1(s) + G_{22}(s)\, R_2(s)$$

The transfer function $G_{11}(s)$ represents the transfer function between output 1 and input 1 when input 2 is zero. Similar definitions can be given to the other transfer functions, $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$.

In general, if a linear system has p inputs and q outputs as shown in Figure 2.5(b), the transfer function between the jth input and the ith output is defined as

$$G_{ij}(s) = \frac{C_i(s)}{R_j(s)}$$

with $R_k(s) = 0$, $k = 1,2, \ldots, p$, $k \neq j$. Note that $G_{ij}(s)$ is defined with only the jth input in effect, whereas the other inputs are set to zero. When all the p inputs are in action, the ith output transform is written as

$$C_i(s) = G_{i1}(s)R_1(s) + G_{i2}(s)R_2(s) + \cdots + G_{ip}(s)R_p(s)$$

In vector-matrix form, the input-output relation of a multivariable system is

$$\mathbf{C}(s) = \mathbf{G}(s)\mathbf{R}(s)$$

where
$$C(s) = \begin{bmatrix} C_1(s) \\ C_2(s) \\ \vdots \\ C_q(s) \end{bmatrix}$$

is the $q \times 1$ transformed output vector,

$$R(s) = \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_p(s) \end{bmatrix}$$

is the $p \times 1$ transformed input vector; and

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & G_{2p}(s) \\ \vdots & \vdots & & \vdots \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qp}(s) \end{bmatrix}$$

is the $q \times p$ transfer function matrix.

2.5.1 Sinusoidal Transfer Function

The steady-state response of a control system to a sinusoidal input is obtained by replacing s with $j\omega$ in the transfer function of the system.

The transfer function obtained by replacing s with $j\omega$ in the original transfer function is called the *sinusoidal transfer function*.

2.6 PROCEDURE FOR DERIVING TRANSFER FUNCTIONS

The following assumptions are made in deriving transfer functions of physical systems.

1. It is assumed that there is no loading, i.e. no power is drawn at the output of the system. If the system has more than one non-loading elements in cascade, then the transfer function of each element can be determined independently, and the overall transfer function of the physical system is determined by multiplying the individual transfer functions. In case of systems consisting of elements which load each other, the overall transfer function should be derived by basic analysis without regard to individual transfer functions.
2. The systems should be approximated by linear, lumped, constant parameter models by making suitable assumptions.

To illustrate point 1 above, let us consider two identical *RC* circuits connected in cascade so that the output from the first circuit is fed as input to the second as shown in Figure 2.6.

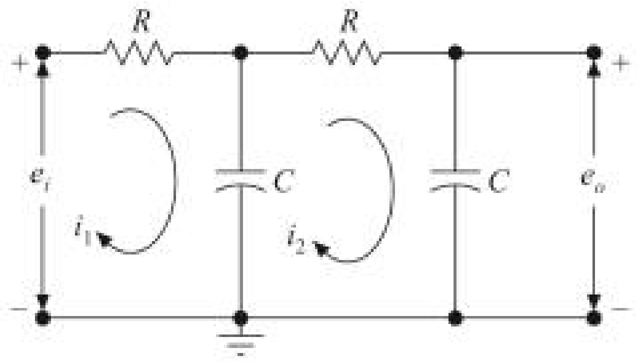


Figure 2.6 RC circuits in cascade.

The describing equations for this system are as follows:

$$\frac{1}{C}\int_{-\infty}^{t}(i_1 - i_2)dt + Ri_1 = e_i \tag{2.3}$$

$$\frac{1}{C}\int_{-\infty}^{t}(i_2 - i_1)dt + Ri_2 = -\frac{1}{C}\int i_2 dt = -e_o \tag{2.4}$$

Taking the Laplace transform of Eqs. (2.3) and (2.4), assuming zero initial conditions, we obtain

$$\frac{1}{Cs}\left[I_1(s) - I_2(s)\right] + RI_1(s) = E_i(s) \tag{2.5}$$

$$\frac{1}{Cs}\left[I_2(s) - I_1(s)\right] + RI_2(s) = -\frac{1}{Cs}I_2(s) = -E_o(s) \tag{2.6}$$

Reorganizing Eq. (2.5), we get

$$I_1(s)\left[R + \frac{1}{Cs}\right] = E_i(s) + \frac{1}{Cs}I_2(s)$$

or

$$I_1(s) = \frac{E_i(s) + \dfrac{1}{Cs}I_2(s)}{R + \dfrac{1}{Cs}} = \frac{CsE_i(s) + I_2(s)}{1 + RCs} \tag{2.7}$$

Reorganizing Eq. (2.6), we get

$$I_2(s)\left[R + \frac{1}{Cs}\right] - \frac{I_1(s)}{Cs} = -E_o(s)$$

Substituting the value of $I_1(s)$ from Eq. (2.7) in the above equation, we get

$$I_2(s)\left[R + \frac{1}{Cs}\right] - \frac{CsE_i(s) + I_2(s)}{Cs(1 + RCs)} = -E_o(s)$$

i.e.

$$CsE_o(s)\left[\frac{1 + RCs}{Cs}\right] - \frac{E_i(s)}{1 + RCs} - \frac{CsE_o(s)}{Cs(1 + RCs)} = -E_o(s) \qquad [\because I_2(s) = CsE_o(s)]$$

or
$$E_o(s)\left[(1+RCs)+1-\frac{1}{1+RCs}\right]=\frac{E_i(s)}{1+RCs}$$

Therefore, the transfer function is

$$\frac{E_o(s)}{E_i(s)}=\frac{1}{\tau^2s^2+3\tau s+1}$$

where $\tau = RC$.

The transfer function of each of the individual *RC* circuits is

$$\frac{\dfrac{1}{Cs}}{R+\dfrac{1}{Cs}}=\frac{1}{1+RCs}=\frac{1}{1+\tau s}$$

If there is no loading, the overall transfer function of the two *RC* circuits connected in cascade is

$$\frac{1}{1+\tau s}\times\frac{1}{1+\tau s}=\frac{1}{\tau^2s^2+2\tau s+1}$$

The difference in the transfer functions is due to the fact that while deriving the transfer function of a single *RC* circuit, it is assumed that the output is unloaded. However, when the input of the second circuit is obtained from the output of the first, a certain amount of energy is drawn from the first circuit and hence its original transfer function is no longer valid. The degree to which the overall transfer function is modified from the product of individual transfer functions depends upon the amount of loading.

Example 2.1 For the mass-spring dashpot system shown in Figure 2.7(a), obtain the transfer function. Also obtain the analogous electrical network based on (a) force-voltage analogy and (b) force-current analogy.

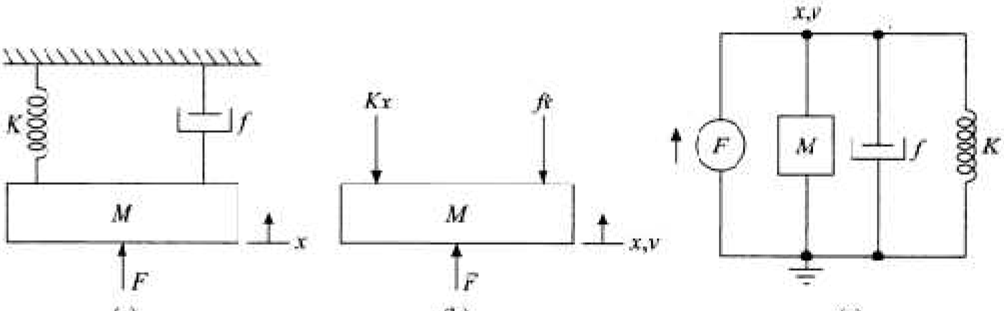


Figure 2.7 Example 2.1: (a) mechanical system, (b) free-body diagram and (c) mechanical network.

Solution: The free-body diagram of the mechanical system of Figure 2.7(a) is shown in Figure 2.7(b). Applying Newton's law of motion that the algebraic sum of the forces acting on a rigid body = mass × acceleration, the equation of motion is

$$F - f\frac{dx}{dt} - Kx = M\frac{d^2x}{dt^2}$$

i.e.

$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx$$

or

$$F = M\frac{dv}{dt} + fv + K\int v\,dt$$

Taking the Laplace transform on both sides and neglecting the initial conditions,

$$F(s) = Ms^2X(s) + fsX(s) + KX(s)$$

$$\therefore \quad \text{Transfer function} = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + fs + K}$$

Alternative way: For the given mechanical system, first draw the mechanical network shown in Figure 2.7(c).

There is only one node. Let the node variable be displacement x or velocity v. One end of all the three elements is connected to ground so the displacement of that end is zero. The other end of the three elements has a displacement x or velocity v. So the describing equation is

$$F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx$$

or

$$F = M\frac{dv}{dt} + fv + K\int v\,dt$$

The analogous equation of the electrical circuit based on force-current analogy is

$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$$

or

$$i = C\frac{de}{dt} + \frac{e}{R} + \frac{1}{L}\int e\,dt$$

and the analogous electrical network is as shown in Figure 2.8(a). It is exactly identical to the mechanical network. Once the mechanical network of a translational system is drawn, the analogous electrical network based on force-current analogy can be easily drawn like this: the structure of the electrical network is exactly the same as the structure of the mechanical network, i.e. series elements in the mechanical network remain as series elements in the electrical network and shunt elements in the mechanical network remain as shunt elements in the electrical network. Just replace the force $F(t)$ by a current source $i(t)$ (= $F(t)$), mass M by capacitance C (= M), friction f by resistance R (= $1/f$), spring K by inductance L (= $1/K$), displacement x by flux ϕ, and velocity v by voltage e. The node equations of the electrical network will be analogous to the node equations of the mechanical network.

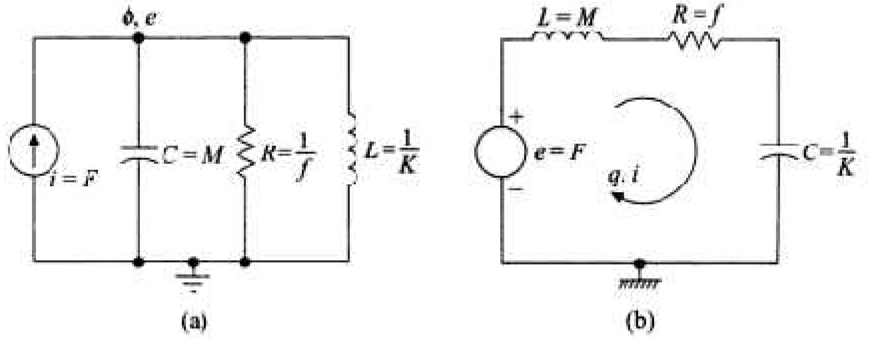


Figure 2.8 Example 2.1: (a) analogous electrical circuit based on F-i analogy and, (b) analogous electrical circuit based on F-v analogy.

The analogous equation of the electrical circuit based on force-voltage analogy is

$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

or

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int i dt$$

and the analogous electrical network is as shown in Figure 2.8(b). Once a mechanical network is available, the analogous electrical network based on force-voltage analogy can be drawn easily like this. In this, the series elements of the mechanical network become analogous shunt elements and the shunt elements of the mechanical network become analogous series elements. Replace the force $F(t)$ by a voltage source $e(t)$ (= $F(t)$), mass M by inductance L (= M), friction f by resistance R (= f), spring K by capacitance C (= $1/K$), displacement x by charge q and velocity v by current $i(t)$. The loop equations will be analogous to the node equations of mechanical network.

Example 2.2 Write the torque equations of the rotational system shown in Figure 2.9(a). Obtain the transfer function $\frac{\theta(s)}{E(s)}$. Also obtain the analogous electrical network based on (a) force-voltage analogy and (b) force-current analogy.

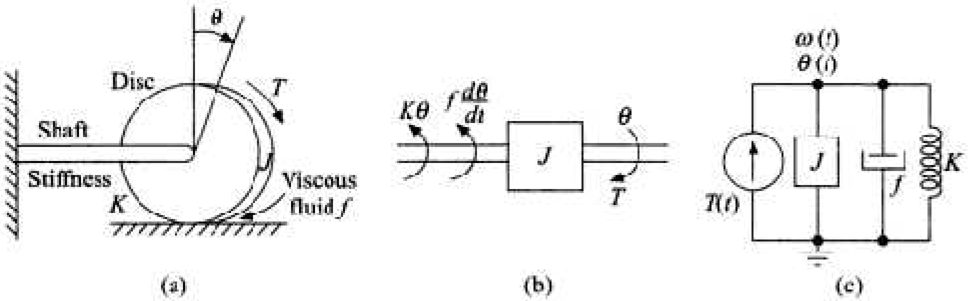


Figure 2.9 Example 2.2: (a) rotational system, (b) free-body diagram and (c) mechanical network.

Solution: The free-body diagram of the mechanical system of Figure 2.9(a) is shown in Figure 2.9(b). Applying Newton's law of motion that the algebraic sum of torques on a body = moment of inertia × angular acceleration, the equation of motion of the system is

$$T - f\frac{d\theta}{dt} - K\theta = J\frac{d^2\theta}{dt^2}$$

i.e.

$$T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K\theta$$

Taking the Laplace transform on both sides

$$T(s) = [Js^2 + fs + K]\,\theta(s)$$

The transfer function of the system is therefore

$$\frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + fs + K}$$

Alternative way: For the given mechanical system, first draw the mechanical network as shown in Figure 2.9(c). One end of source (T) and moment of inertia (J) is always to be connected to ground. In the given system, T is applied on the body with moment of inertia J. So the other ends of T and J are to be joined. In Figure 2.9(a), one end of f is connected to ground and the other end is connected to J. The same is done in drawing the network. In the network, there is only one node. Let the node variable be θ or ω. So the describing equation is

$$T = J\frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K\theta$$

or

$$T = J\frac{d\omega}{dt} + f\omega + K\int_{-\infty}^{t}\omega\, dt$$

The analogous equation of the electrical circuit based on torque-current analogy is

$$i = C\frac{d^2\phi}{dt^2} + \frac{1}{R}\frac{d\phi}{dt} + \frac{1}{L}\phi$$

or

$$i = C\frac{de}{dt} + \frac{1}{R}e + \frac{1}{L}\int_{-\infty}^{t} e\, dt$$

and the analogous electrical circuit is shown in Figure 2.10(a). It is exactly identical to the mechanical network. Once the mechanical network of a rotational system is drawn, the analogous electrical network based on torque-current analogy can be easily drawn like this: the structure of the electrical network is exactly the same as the structure of the mechanical network, i.e. series elements in the mechanical network remain as series elements in the electrical network and shunt elements in the mechanical network remain as shunt elements in the electrical network. So the node equations of the electrical network will be analogous to the node equations of mechanical network. Just replace the torque $T(t)$ by current source $i(t)$ (= $T(t)$), inertia J by capacitance C (= J), friction f by resistance R (= $1/f$), spring K by inductance L (= $1/K$), angular displacement θ by ϕ and angular velocity ω by voltage e.

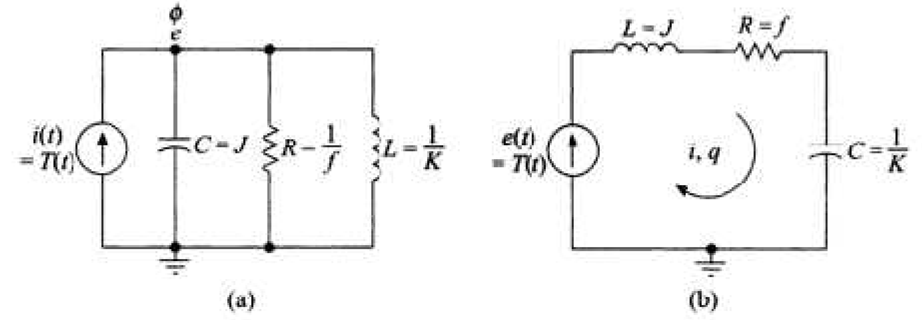


Figure 2.10 Example 2.2: (a) analogous electrical network based on torque-current analogy and (b) analogous electrical network based on torque-voltage analogy.

The analogous equation of the electrical circuit based on torque-voltage analogy is

$$e = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

or

$$e = L\frac{di}{dt} + Ri + \frac{1}{C}\int_{-\infty}^{t} i\,dt$$

and the analogous electrical circuit is shown in Figure 2.10(b). Once the mechanical network of a rotational system is drawn, the analogous electrical network based on force-voltage analogy can be easily drawn like this. In this, the series elements of the mechanical network become analogous shunt elements and the shunt elements of the mechanical network become analogous series elements. Replace torque $T(t)$ by a voltage source $e(t)$ (= $T(t)$), inertia J by inductance L (= J), friction f by resistance $R(= f)$, spring K by capacitance C (= $1/K$), angular displacement θ by charge q, and angular velocity ω by $i(t)$. So the loop equations of the electrical network will be analogous to the node equations of the mechanical network.

Example 2.3 Write the differential equations for the mechanical system shown in Figure 2.11(a).

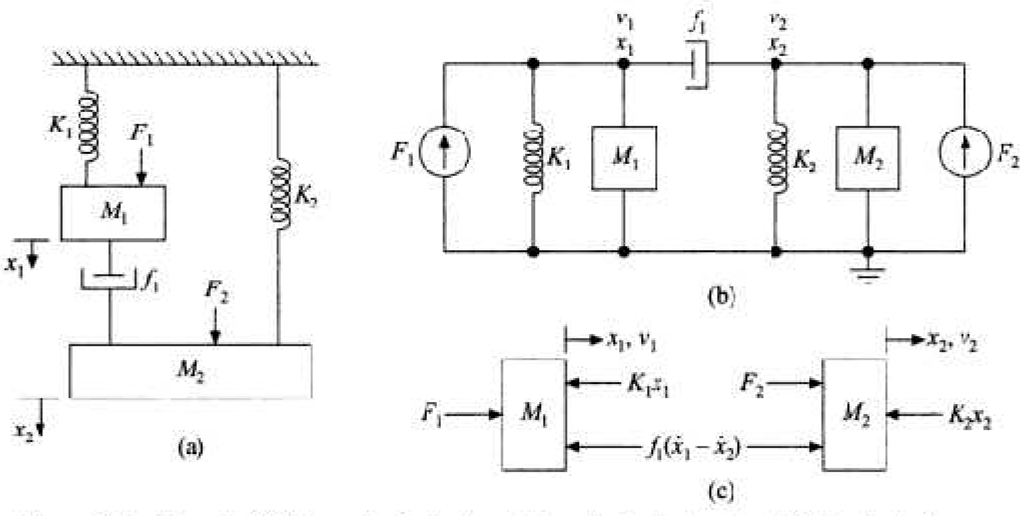


Figure 2.11 Example 2.3: (a) mechanical system, (b) mechanical network and (c) free body diagram.

Solution: The mechanical network for the given mechanical translational system is shown in Figure 2.11(b). The free body diagram is shown in Figure 2.11(c). The differential equations governing the behaviour of that system are given as follows. The node variables may be displacements or velocities.

The differential equations for the system are as follows:

$$M_2\frac{d^2x_2}{dt^2}+f_1\left(\frac{dx_2}{dt}-\frac{dx_1}{dt}\right)+K_2x_2=F_2(t)$$

or

$$M_2\frac{dv_2}{dt}+f_1(v_2-v_1)+K_2\int v_2\,dt=F_2(t)$$

$$M_1\frac{d^2x_1}{dt^2}+f_1\left(\frac{dx_1}{dt}-\frac{dx_2}{dt}\right)+K_1x_1=F_1(t)$$

or

$$M_1\frac{dv_1}{dt}+f_1(v_1-v_2)+K_1\int v_1\,dt=F_1(t)$$

Example 2.4 Find the transfer function $\dfrac{X_o(s)}{X_i(s)}$ for the system shown in Figure 2.12(a).

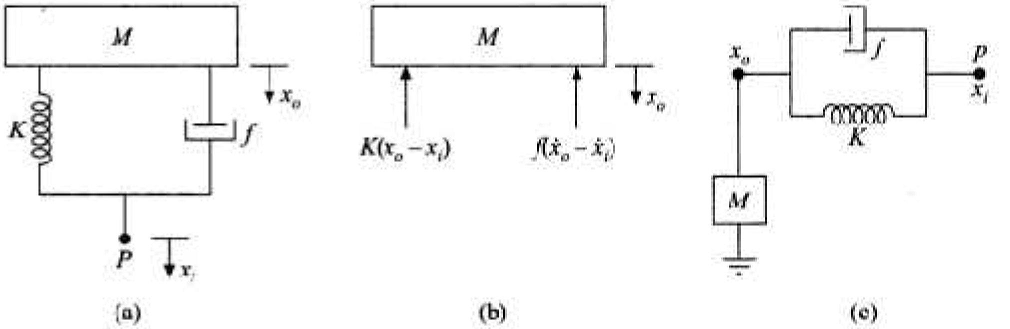


Figure 2.12 Example 2.4: (a) mechanical system, (b) free-body diagram and (c) mechanical network.

Solution: The free-body diagram is shown in Figure 2.12(b). The mechanical network is shown in Figure 2.12(c). No external force is applied. The displacements are due to initial conditions. The differential equation governing the behaviour of the system is

$$M\frac{d^2x_o}{dt^2}+f\left(\frac{dx_o}{dt}-\frac{dx_i}{dt}\right)+K(x_o-x_i)=0$$

Taking the Laplace transform on both sides

$$Ms^2X_o(s)+fs\,[X_o(s)-X_i(s)]+K\,[X_o(s)-X_i(s)]=0$$

i.e.

$$[Ms^2+fs+K]\,X_o(s)=(fs+K)X_i(s)$$

The transfer function is therefore

$$\frac{X_o(s)}{X_i(s)}=\frac{fs+K}{Ms^2+fs+K}$$

Example 2.5 Obtain the differential equations for the mechanical system shown in Figure 2.13(a).

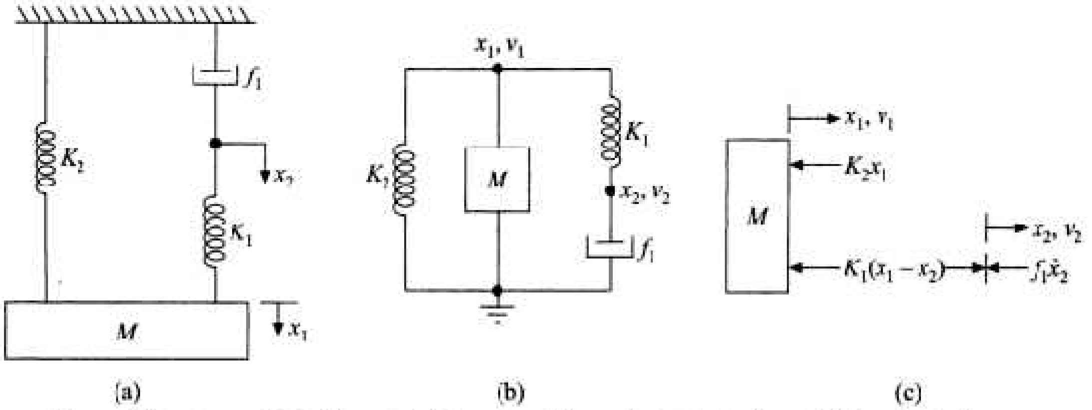


Figure 2.13 Example 2.5: (a) mechanical system, (b) mechanical network and (c) free body diagram.

Solution: The mechanical network is shown in Figure 2.13(b). The free body diagram is shown in Figure 2.13(c). The differential equations for the given mechanical system are

$$M\frac{d^2x_1}{dt^2} + K_2x_1 + K_1(x_1 - x_2) = 0$$

or

$$M\frac{dv_1}{dt} + K_2\int v_1\,dt + K_1\int (v_1 - v_2)\,dt = 0$$

and

$$K_1(x_2 - x_1) + f_1\frac{dx_2}{dt} = 0$$

or

$$K_1\int (v_2 - v_1)\,dt + f_1v_2 = 0$$

Example 2.6 Consider the mechanical system shown in Figure 2.14(a). Suppose that the system is set into motion by unit impulse force. Find the resulting oscillation. Assume that the system is at rest initially.

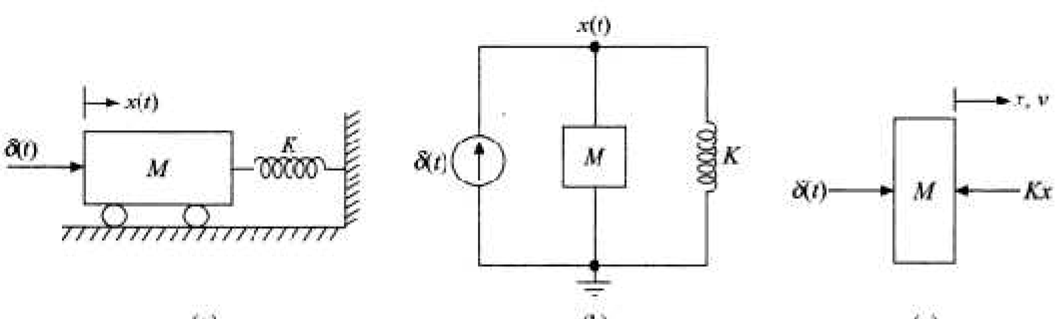


Figure 2.14 Example 2.6: (a) mechanical system, (b) mechanical network and (c) free body diagram.

Solution: The mechanical network is shown in Figure 2.14(b). The free body diagram is shown in Figure 2.14(c).

The differential equation of the system is

$$\delta(t) = Kx + M\frac{d^2x}{dt^2}$$

Taking the Laplace transform on both sides and neglecting the initial conditions.

$$1 = (Ms^2 + K)X(s)$$

i.e.
$$X(s) = \frac{1}{Ms^2 + K} = \frac{1}{M\left(s^2 + \frac{K}{M}\right)} = \frac{1}{M}\left[\frac{\sqrt{K/M}}{s^2 + \left(\sqrt{K/M}\right)^2}\right]\frac{1}{\sqrt{K/M}}$$

Taking the inverse Laplace transform

$$x(t) = \frac{1}{\sqrt{MK}}\sin\left(\sqrt{K/M}\right)t$$

The resulting oscillation frequency is $\omega = \sqrt{K/M}$ rad/s.

Example 2.7 Assume that the cart in Figure 2.15(a) is standing still for $t < 0$, $u(t)$ is the displacement of the cart and $y(t)$ is the output. Obtain the transfer function of the system.

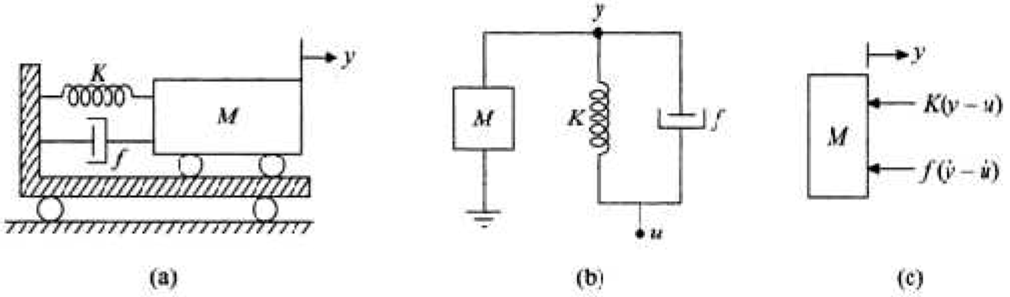


Figure 2.15 Example 2.7: (a) mechanical system, (b) mechanical network and (c) free body diagram.

Solution: The mechanical network of the system is shown in Figure 2.15(b). The free body diagram is shown in Figure 2.15(c). The differential equation of the system is

$$M\frac{d^2y}{dt^2} + f\left(\frac{dy}{dt} - \frac{du}{dt}\right) + K(y - u) = 0$$

Taking the Laplace transform on both sides and neglecting the initial conditions

$$Ms^2Y(s) + fs[Y(s) - U(s)] + K[Y(s) - U(s)] = 0$$

i.e.
$$[Ms^2 + fs + K]\,Y(s) = [fs + K]\,U(s)$$

Therefore, the transfer function of the mechanical system is

$$\frac{Y(s)}{U(s)} = \frac{fs + K}{Ms^2 + fs + K}$$

Example 2.8 Write the differential equations for the mechanical system shown in Figure 2.16. Obtain the analogous electrical networks based on (a) force-current analogy and (b) force-voltage analogy.

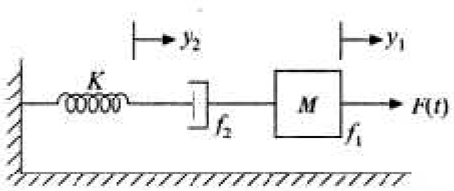
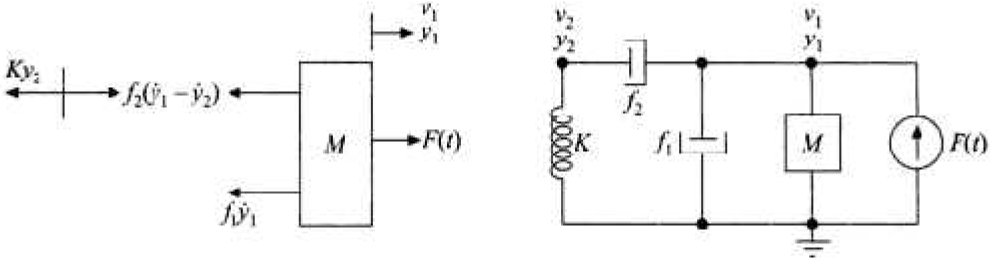


Figure 2.16 Example 2.8: Mechanical system.

Solution: The free-body diagram is shown in Figure 2.17(a). The mechanical network of the system is shown in Figure 2.17(b).



(a) (b)

Figure 2.17 Example 2.8: (a) free-body diagram, and (b) mechanical network.

The differential equations describing the behaviour of the given mechanical system are

$$F(t) = M\frac{d^2 y_1}{dt^2} + f_1\frac{dy_1}{dt} + f_2\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right)$$

or

$$F(t) = M\frac{dv_1}{dt} + f_1 v_1 + f_2(v_1 - v_2)$$

and

$$0 = f_2\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) + Ky_2$$

or

$$0 = f_2(v_2 - v_1) + K\int_{-\infty}^{t} v_2\, dt$$

The electrical networks based on force-current and force-voltage analogies are shown in Figure 2.18(a) and Figure 2.18(b) respectively and the corresponding differential equations are also given as follows:

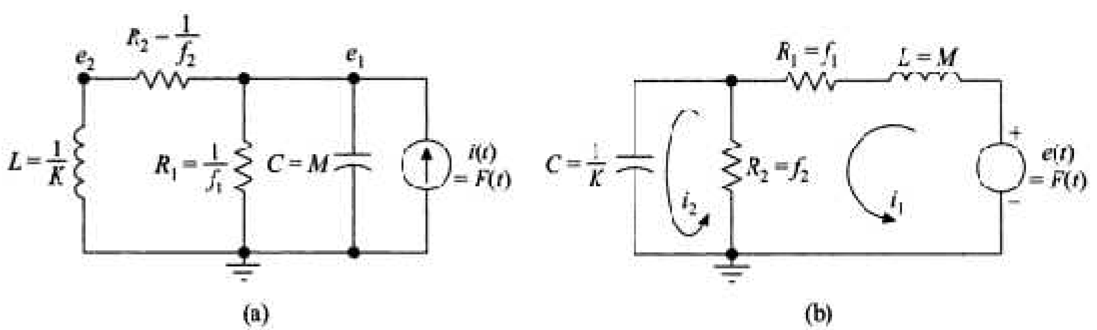


Figure 2.18 Example 2.8: (a) electrical network based on force-current analogy and (b) electrical network based on force-voltage analogy.

$$i(t) = C\frac{de_1}{dt} + \frac{e_1}{R_1} + \frac{e_1 - e_2}{R_2} \qquad e(t) = L\frac{di_1}{dt} + R_1 i_1 + R_2(i_1 - i_2)$$

$$\frac{e_2 - e_1}{R_2} + \frac{1}{L}\int e_2\, dt = 0 \qquad 0 = R_2(i_2 - i_1) + \frac{1}{C}\int i_2\, dt$$

Example 2.9 Write the differential equations for the mechanical system shown in Figure 2.19(a). Also draw the analogous electrical circuit based on force-current analogy.

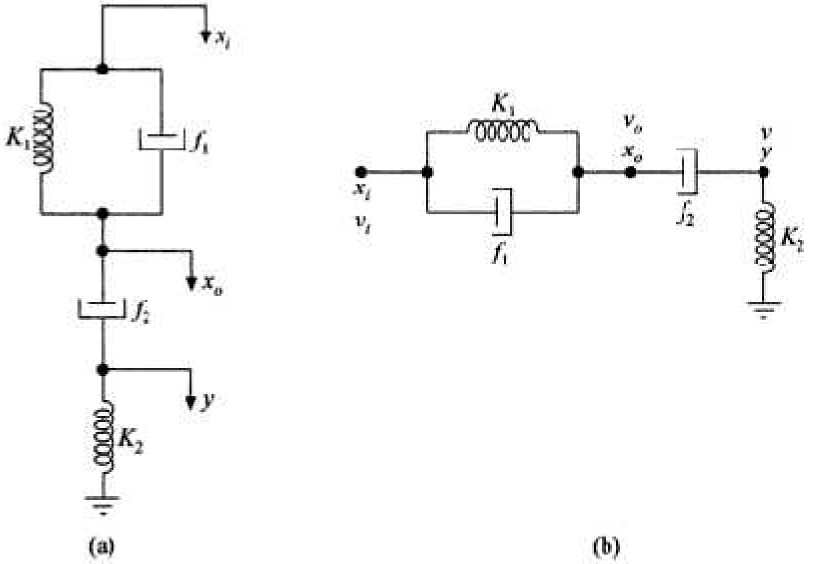


Figure 2.19 Example 2.9: (a) mechanical system and (b) mechanical network.

Solution: The mechanical network is shown in Figure 2.19(b). No external force is acting on the system and the displacements are due to initial conditions only.

The differential equations for the system are

$$K_1(x_o - x_i) + f_1\left(\frac{dx_o}{dt} - \frac{dx_i}{dt}\right) + f_2\left(\frac{dx_o}{dt} - \frac{dy}{dt}\right) = 0$$

or
$$K_1\int (v_o - v_i)\,dt + f_1(v_o - v_i) + f_2(v_o - v) = 0$$

and
$$f_2\left(\frac{dy}{dt} - \frac{dx_o}{dt}\right) + K_2 y = 0$$

or
$$f_2(v - v_o) + K_2\int v\,dt = 0$$

The analogous electrical circuit based on force-current analogy is shown in Figure 2.20.

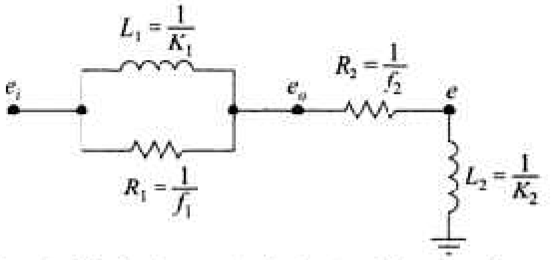


Figure 2.20 Example 2.9: Analogous electrical network based on force-current analogy.

The differential equations for the network of Figure 2.20 are as follows:

$$\frac{e_o - e_i}{R_1} + \frac{1}{L_1}\int (e_o - e_i)\,dt + \frac{e_o - e}{R_2} = 0$$

$$\frac{e - e_o}{R_2} + \frac{1}{L_2}\int e\,dt = 0$$

Example 2.10 Write the differential equations governing the behaviour of the mechanical system shown in Figure 2.21. Also obtain the analogous electrical circuits based on (a) force-current analogy and (b) force-voltage analogy. Also find the transfer function $X_1(s)/F(s)$.

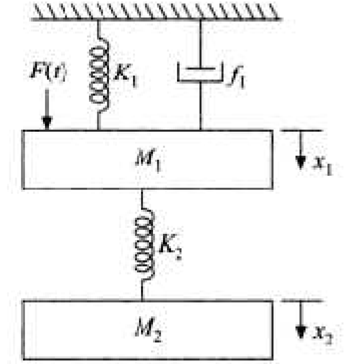


Figure 2.21 Example 2.10: Mechanical system.

Solution: The differential equations governing the behaviour of the mechanical system can be written using the free-body diagram or the mechanical network. Just for illustration both the free-body diagram and mechanical network are drawn as shown in Figure 2.22(a) and Figure 2.22(b) respectively.

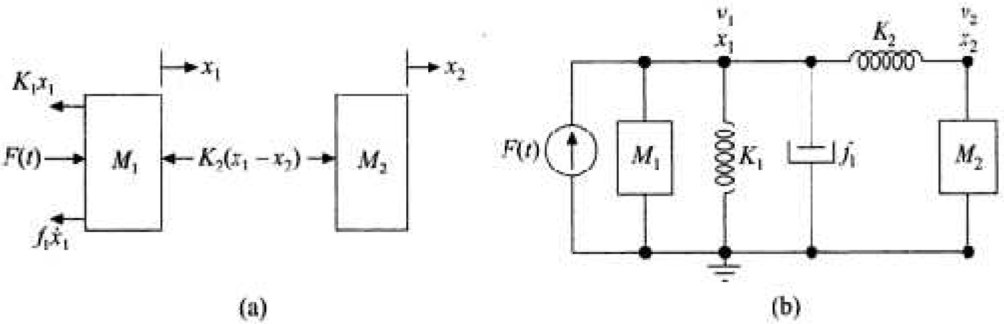


Figure 2.22 Example 2.10: (a) free-body diagram and (b) mechanical network.

The differential equations describing the behaviour of the mechanical system are as follows:

$$F(t) = M_1 \frac{d^2 x_1}{dt^2} + f_1 \frac{dx_1}{dt} + K_1 x_1 + K_2 (x_1 - x_2)$$

i.e.
$$F(t) = M_1 \frac{dv_1}{dt} + f_1 v_1 + K_1 \int v_1 dt + K_2 \int (v_1 - v_2) dt$$

$$M_2 \frac{d^2 x_2}{dt^2} + K_2 (x_2 - x_1) = 0$$

i.e.
$$M_2 \frac{dv_2}{dt} + K_2 \int (v_2 - v_1)\, dt = 0$$

The analogous electrical network based on force-current analogy is shown in Figure 2.23.

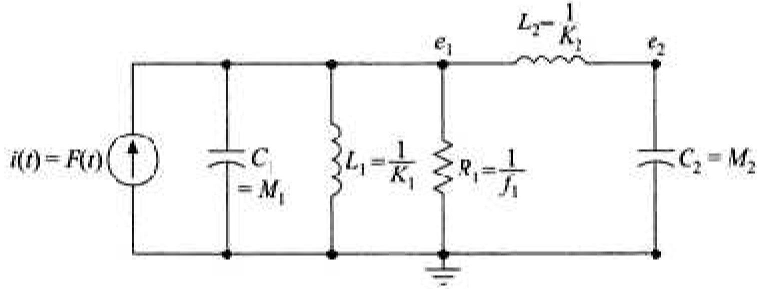


Figure 2.23 Example 2.10: Analogous electrical network based on force-current analogy.

The analogous electrical equations are as follows:

$$i(t) = C_1 \frac{de_1}{dt} + \frac{e_1}{R_1} + \frac{1}{L_1} \int e_1\, dt + \frac{1}{L_2} \int (e_1 - e_2)\, dt$$

$$C_2 \frac{de_2}{dt} + \frac{1}{L_2} \int (e_2 - e_1)\, dt = 0$$

The analogous electrical network based on force-voltage analogy is shown in Figure 2.24.

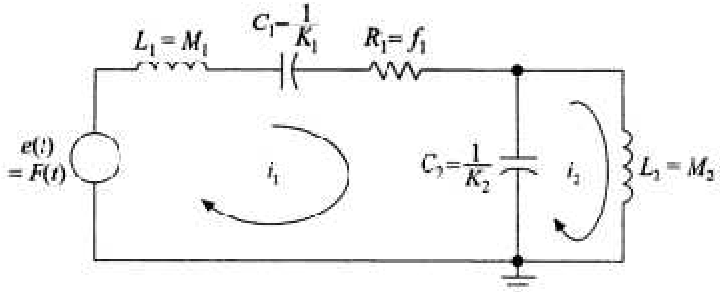


Figure 2.24 Example 2.10: Analogous electrical network based on force-voltage analogy.

The analogous electrical equations are as follows:

$$e(t) = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1} \int i_1 dt + \frac{1}{C_2} \int (i_1 - i_2)\, dt$$

$$0 = L_2 \frac{di_2}{dt} + \frac{1}{C_2} \int (i_2 - i_1)\, dt$$

Example 2.11 For the mechanical system shown in Figure 2.25, write the differential equations describing its behaviour. Write the analogous electrical equations based on force-voltage analogy, and force-current analogy, and draw the corresponding networks. Also draw the mechanical network and obtain the transfer function $X(s)/F(s)$.

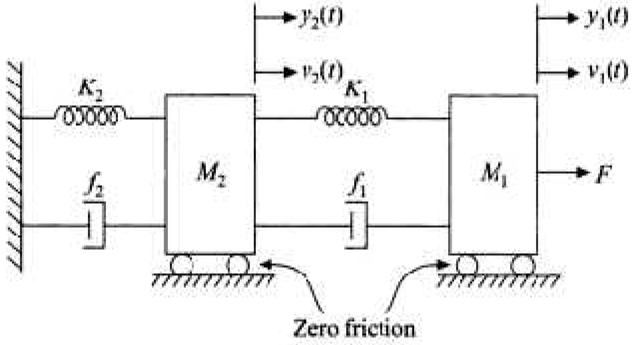


Figure 2.25 Example 2.11: Mechanical system.

Solution: The free-body diagram of the mechanical system is shown in Figure 2.26

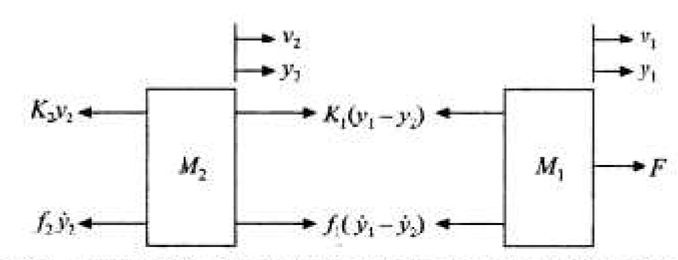


Figure 2.26 Example 2.11: Free-body diagram for mechanical system of Figure 2.25.

The describing equations are as follows:

$$F - f_1\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) - K_1(y_1 - y_2) = M_1\frac{d^2y_1}{dt^2}$$

i.e.

$$F = M_1\frac{d^2y_1}{dt^2} + f_1\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K_1(y_1 - y_2)$$

$$K_1(y_1 - y_2) + f_1\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) - K_2y_2 - f_2\frac{dy_2}{dt} = M_2\frac{d^2y_2}{dt^2}$$

i.e.

$$f_1\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) + K_1(y_2 - y_1) + K_2y_2 + f_2\frac{dy_2}{dt} + M_2\frac{d^2y_2}{dt^2} = 0$$

The describing equations in terms of velocities are as follows:

$$F = M_1\frac{dv_1}{dt} + f_1(v_1 - v_2) + K_1\int(v_1 - v_2)\,dt$$

$$f_1(v_2 - v_1) + K_1\int(v_2 - v_1)\,dt + f_2v_2 + K_2\int v_2\,dt + M_2\frac{dv_2}{dt} = 0$$

The equations of the analogous electrical network based on force-current analogy are as follows:

$$i(t) = C_1\frac{de_1}{dt} + \frac{1}{R_1}(e_1 - e_2) + \frac{1}{L_1}\int(e_1 - e_2)\,dt$$

$$\frac{1}{R_1}(e_2 - e_1) + \frac{1}{L_1}\int(e_2 - e_1)\,dt + \frac{1}{R_2}e_2 + \frac{1}{L_2}\int e_2dt + C_2\frac{de_2}{dt} = 0$$

The corresponding electrical network is shown in Figure 2.27.

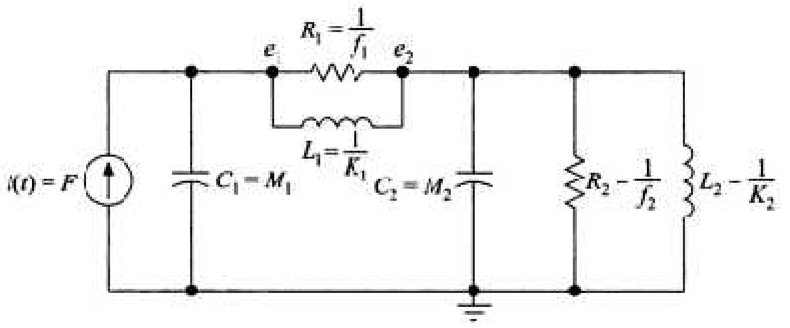


Figure 2.27 Example 2.11: Analogous electrical network based on force-current analogy.

The equations of the analogous electrical network based on force-voltage analogy are as follows:

$$e = L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_2)\, dt$$

and

$$\frac{1}{C_1}\int (i_2 - i_1)\, dt + R_1(i_2 - i_1) + R_2 i_2 + \frac{1}{C_2}\int i_2 dt + L_2 \frac{di_2}{dt} = 0$$

The corresponding electrical network is shown in Figure 2.28.

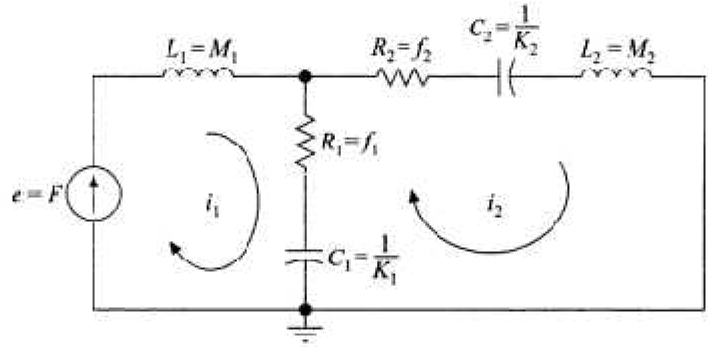


Figure 2.28 Example 2.11: Analogous electrical network based on force-voltage analogy.

The mechanical network for the given mechanical system is shown in Figure 2.29.

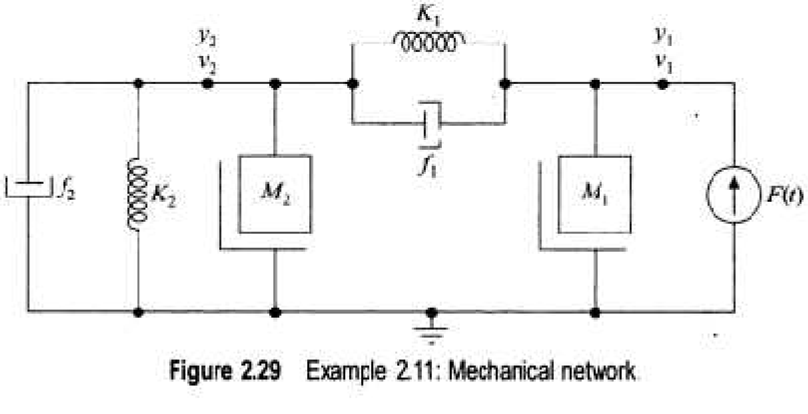


Figure 2.29 Example 2.11: Mechanical network.

The describing equations at nodes 1 and 2 of Figure 2.29 are as follows:

$$M_2 \frac{d^2 y_2}{dt^2} - f_2 \frac{dy_2}{dt} + K_2 y_2 + f_1\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) + K_1(y_2 - y_1) = 0$$

$$F(t) = M_1 \frac{d^2 y_1}{dt^2} + f_1\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) + K_1(y_1 - y_2)$$

To obtain the transfer function, take the Laplace transform of the describing equations of the mechanical system neglecting the initial conditions.

$$F(s) = M_1 s^2 Y_1(s) + f_1 s[Y_1(s) - Y_2(s)] + K_1[Y_1(s) - Y_2(s)]$$

i.e.
$$F(s) = [M_1 s^2 + f_1 s + K_1]\, Y_1(s) - [f_1 s + K_1] Y_2(s) \quad \text{(i)}$$

$$M_2 s^2 Y_2(s) + f_2 s Y_2(s) + K_2 Y_2(s) + f_1 s[Y_2(s) - Y_1(s)] + K_1[Y_2(s) - Y_1(s)] = 0$$

$$[M_2 s^2 + (f_2 + f_1)s + (K_1 + K_2)] Y_2(s) - [f_1 s + K_1] Y_1(s) = 0$$

$\therefore$
$$Y_2(s) = \frac{[f_1 s + K_1] Y_1(s)}{M_2 s^2 + (f_2 + f_1)s + (K_1 + K_2)} \quad \text{(ii)}$$

Substituting the value of $Y_2(s)$ from Eq. (ii) in Eq. (i), we get

$$F(s) = [M_1 s^2 + f_1 s + K_1]\, Y_1(s) - \frac{(f_1 s + K_1)(f_1 s + K_1) Y_1(s)}{M_2 s^2 + (f_2 + f_1)s + (K_2 + K_1)}$$

Therefore, the transfer function is

$$\frac{Y_1(s)}{F(s)} = \frac{M_2 s^2 + (f_2 + f_1)s + (K_2 + K_1)}{(M_1 s^2 + f_1 s + K_1)(M_2 s^2 + (f_2 + f_1)s + (K_2 + K_1)) - (f_1 s + K_1)^2}$$

i.e.
$$\frac{Y_1(s)}{F(s)} = \frac{M_2 s^2 + (f_2 + f_1)s + (K_2 + K_1)}{M_1 M_2 s^4 + (M_1 f_1 + M_1 f_2 + M_2 f_1)s^3 + (M_1 K_2 + M_1 K_1 + M_2 K_1 + f_1 f_2)s^2 + (f_1 K_2 + f_2 K_1)s - K_1^2}$$

Example 2.12 Write the differential equations governing the behaviour of the mechanical system shown in Figure 2.30(a).

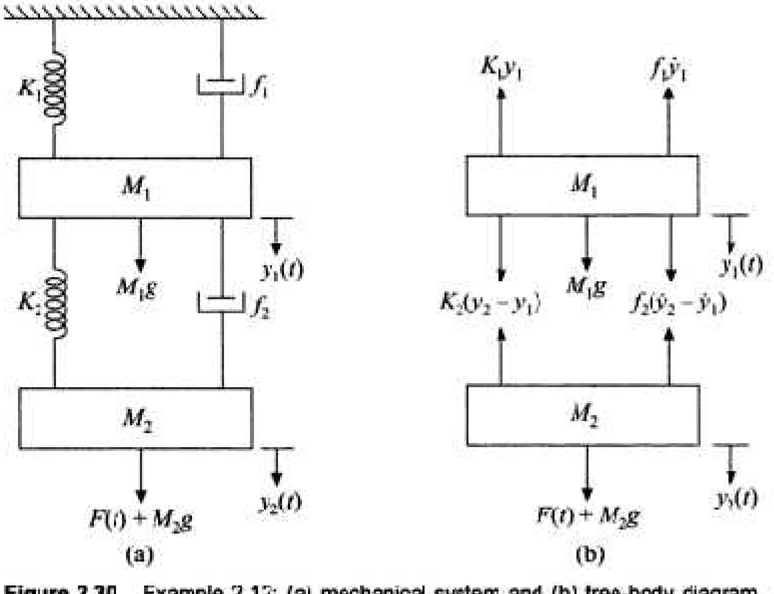


Figure 2.30 Example 2.12: (a) mechanical system and (b) free-body diagram.

Solution: The free-body diagram is shown in Figure 2.33(a). The mechanical network is shown in Figure 2.33 (b). The describing equations can be written based on any one of them.

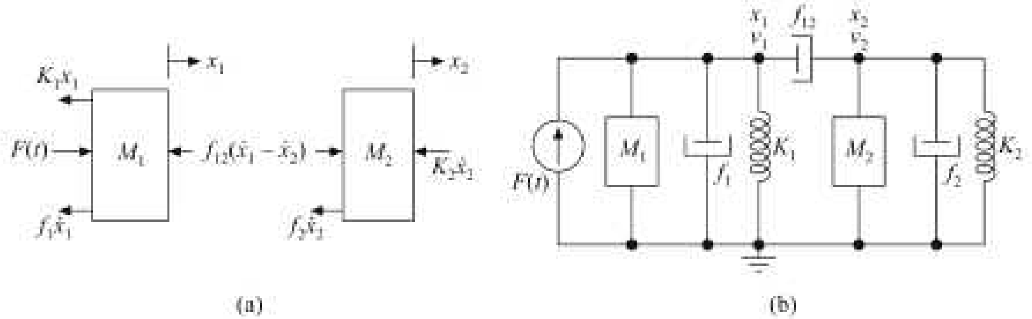


Figure 2.33 Example 2.13: (a) free-body diagram and (b) mechanical network.

The free-body diagram is drawn like this: $F(t)$ is the external force applied on mass M_1 to move it by a distance x_1 from left to right. So direction of $F(t)$ is from left to right. All other forces acting on M_1 will oppose this motion. So the directions of $f_1\dot{x}_1$, K_1x_1 and $f_{12}(\dot{x}_1 - \dot{x}_2)$ are opposite to that of $F(t)$. So they act from right to left. f_{12} is connecting M_1 and M_2. Therefore $f_{12}(\dot{x}_1 - \dot{x}_2)$ acts from left to right on M_2. This will cause a displacement of x_2 on M_2. All other forces at M_2, i.e $f_2\dot{x}_2$ and K_2x_2 act in a direction opposite to that of $f_{12}(\dot{x}_1 - \dot{x}_2)$, i.e. from right to left.

The mechanical network is drawn as follows. There are two masses M_1 and M_2. One end of each mass is to be connected to ground. One end of $F(t)$ is to be connected to ground. The second end of $F(t)$ is to be connected to the second end of M_1. $f_1(t)$ is between M_1 and ground. So one end of f_1 is connected to ground and the other end is connected to the second end of M_1. K_1 is between M_1 and ground. So one end of K_1 is connected to ground and the other end is connected to the second end of M_1.One end of M_2 is grounded. f_{12} is between the second ends of M_1 and M_2. K_2 and f_2 are between M_2 and ground. So one end of K_2 and f_2 is connected to ground and the other ends of K_2 and f_2 are connected to the second end of M_2. The mechanical network is shown in Figure 2.33(b). The displacements x_1 and x_2 or velocities v_1 and v_2 are the node variables at the second ends of M_1 and M_2 respectively.

The differential equations describing the behaviour of the mechanical system are as follows:

$$F(t) = M_1\frac{d^2x_1}{dt^2} + f_1\frac{dx_1}{dt} + K_1x_1 + f_{12}\left(\frac{dx_1}{dt} - \frac{dx_2}{dt}\right)$$

or
$$F(t) = M_1\frac{dv_1}{dt} + f_1v_1 + K_1\int v_1dt + f_{12}(v_1 - v_2)$$

$$0 = M_2\frac{d^2x_2}{dt^2} + f_2\frac{dx_2}{dt} + K_2x_2 + f_{12}\left(\frac{dx_2}{dt} - \frac{dx_1}{dt}\right)$$

or
$$0 = M_2\frac{dv_2}{dt} + f_2v_2 + K_2\int v_2dt + f_{12}(v_2 - v_1)$$

Example 2.14 Write the differential equations governing the behaviour of the mechanical translational system shown in Figure 2.34(a).

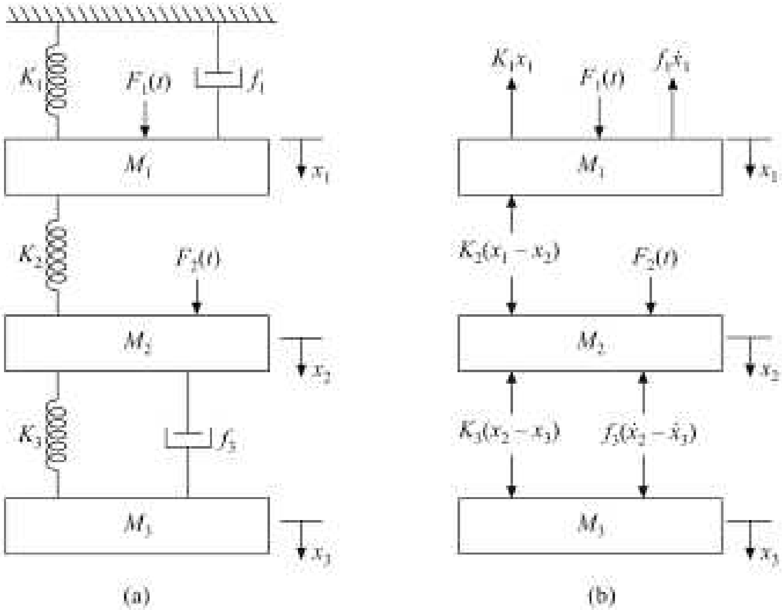


Figure 2.34 Example 2.14: (a) mechanical system and (b) free-body diagram.

Solution: The free-body diagram of the system is shown in Figure 2.34(b). The mechanical network for the given mechanical system is shown in Figure 2.35.

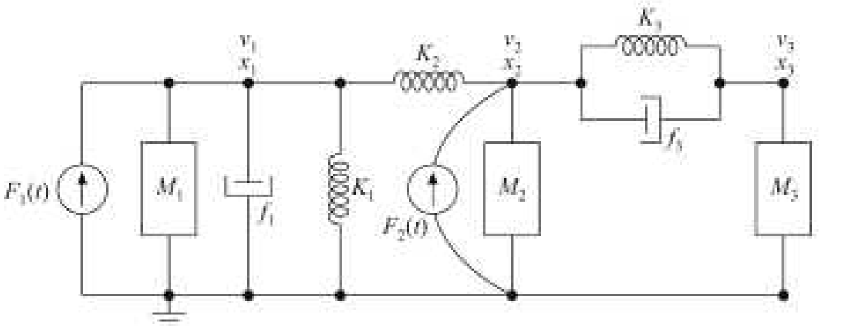


Figure 2.35 Example 2.14: Mechanical network for the mechanical system of Figure 2.34(a).

The differential equations governing the behaviour of the system are as follows:

$$F_1(t) = M_1 \frac{d^2x_1}{dt^2} + f_1 \frac{dx_1}{dt} + K_1x_1 + K_2(x_1 - x_2)$$

or $$F_1(t) = M_1\frac{dv_1}{dt} + f_1v_1 + K_1\int v_1 dt + K_2\int (v_1 - v_2)dt$$

$$F_2(t) = M_2\frac{d^2x_2}{dt^2} + K_2(x_2 - x_1) + f_3\left(\frac{dx_2}{dt} - \frac{dx_3}{dt}\right) + K_3(x_2 - x_3)$$

or $$F_2(t) = M_2\frac{dv_2}{dt} + K_2\int (v_2 - v_1)dt + f_3(v_2 - v_3) + K_3\int (v_2 - v_3)dt$$

$$0 = M_3\frac{d^2x_3}{dt^2} + f_3\left(\frac{dx_3}{dt} - \frac{dx_2}{dt}\right) + K_3(x_3 - x_2)$$

or $$0 = M_3\frac{dv_3}{dt} + f_3(v_3 - v_2) + K_3\int (v_3 - v_2)dt$$

Example 2.15 Find the transfer function $X(s)/E(s)$ for the electro-mechanical system shown in Figure 2.36.

[*Hint:* For a simplified analysis, assume that the coil has a back emf $e_b = K_b\, dx/dt$ and the coil current i produces a force $F_C = K_2\, i$ on the mass M.]

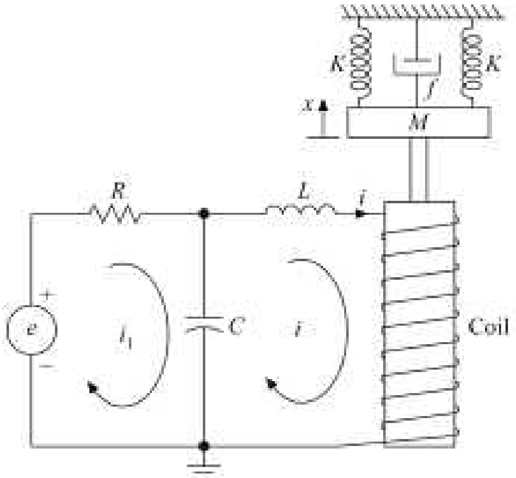


Figure 2.36 Example 2.15: Electromechanical system.

Solution: Let i_1 be the current in loop1 and let i be the current in the loop having the coil. Writing the KVL equations around the loops,

$$e(t) = Ri_1(t) + \frac{1}{C}\int [i_1(t) - i(t)]\, dt$$

and $$L\frac{di}{dt} + e_b(t) + \frac{1}{C}\int [i(t) - i_1(t)]\, dt = 0$$

Taking the Laplace transform on both sides,

$$E(s) = RI_1(s) + \frac{1}{Cs}[I_1(s) - I(s)]$$

i.e. $$E(s) = \left(R + \frac{1}{Cs}\right)I_1(s) - \frac{1}{Cs}I(s)$$

$$= \frac{(RCs+1)I_1(s) - I(s)}{Cs}$$

and $$LsI(s) + E_b(s) + \frac{1}{Cs}[I(s) - I_1(s)] = 0$$

i.e. $$\left(Ls + \frac{1}{Cs}\right)I(s) + K_b sX(s) - \frac{1}{Cs}I_1(s) = 0 \quad \left[e_b = K_b\frac{dx}{dt}, \;\therefore\; E_b(s) = K_b sX(s)\right]$$

$\therefore$ $$I_1(s) = (LCs^2+1)I(s) + K_b Cs^2 X(s)$$

$\therefore$ $$E(s) = \frac{(RCs+1)[(LCs^2+1)I(s) + K_b Cs^2 X(s)] - I(s)}{Cs}$$

$$= I(s)\frac{(RLC^2s^3 + LCs^2 + RCs + 1 - 1) + (RC^2K_b s^3 + K_b Cs^2)X(s)}{Cs}$$

$$= I(s)(RLCs^2 + Ls + R) + (K_b RCs^2 + K_b s)X(s) \qquad \text{(i)}$$

Given $F_C = K_2 i$

$\therefore$ $F_C(s) = K_2 I(s)$

Writing the differential equation governing the behaviour of the mechanical system,

$$F_C = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + Kx + Kx$$

$\therefore$ $$F_C(s) = [Ms^2 + fs + 2K]X(s) = K_2 I(s)$$

$\therefore$ $$I(s) = \frac{Ms^2 + fs + 2K}{K_2}X(s) \qquad \text{(ii)}$$

Substituting the value of $I(s)$ from Eq. (ii) in Eq. (i), we get

$$E(s) = (RLCs^2 + Ls + R)\left(\frac{Ms^2 + fs + 2K}{K_2}\right)X(s) + (K_b RCs^2 + K_b s)X(s)$$

$$= \left(\frac{\begin{array}{r} RLCMs^4 + RLCfs^3 + 2KRLCs^2 + LMs^3 + Lfs^2 + 2KLs + RMs^2 \\ + Rfs + 2RK + K_2K_bRCs^2 + K_2K_bs \end{array}}{K_2} \right) X(s)$$

Therefore, the transfer function is

$$\frac{X(s)}{E(s)} = \frac{K_2}{\begin{array}{r} RLCMs^4 + (RLCf + LM)s^3 + (2KRLC + Lf + RM + K_2K_bRC)s^2 \\ + (2KL + Rf + K_2K_b)s + 2RK \end{array}}$$

Example 2.16 Obtain the transfer function of the mechanical system shown in Figure 2.37.

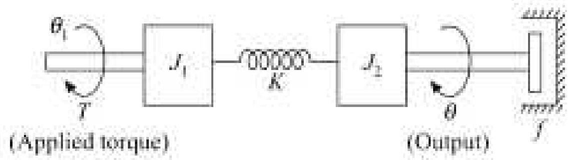


Figure 2.37 Example 2.16: Mechanical system.

Solution: The output is the angular displacement θ of shaft on output side. Let θ_1 be the angular displacement of shaft on input side. The free-body diagram is shown in Figure 2.38(a). The mechanical network corresponding to the given mechanical rotational system is drawn as shown in Figure 2.38(b). One end of J_1 is to be connected to ground. One end of torque source is to be grounded. The other end of T is connected to the other end of J_1 at the node 1. Let this node variable be θ_1 or ω_1. One end of J_2 is grounded, and K is connected between the two free ends of J_2 and J_1. One end of f is grounded, the other end of f is joined to the free end of J_2. Let θ, ω be that node variable.

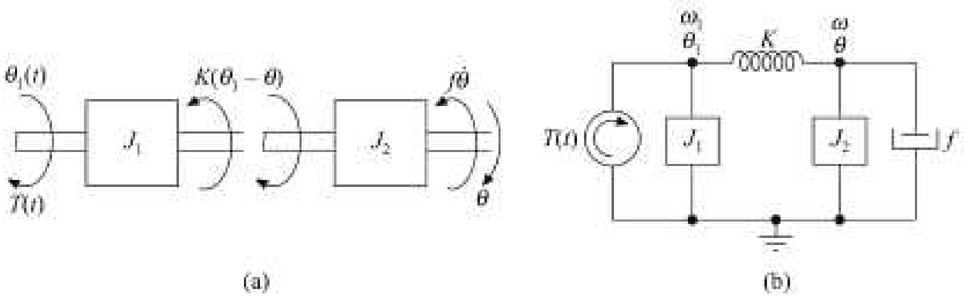


Figure 2.38 Example 2.16: (a) free-body diagram and (b) mechanical network.

The differential equations describing the behaviour of the system in Figure 2.37 are as follows:

$$T = J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta) \quad \text{or} \quad T = J_1 \frac{d\omega_1}{dt} + K\int(\omega_1 - \omega)\,dt \qquad \text{(i)}$$

and
$$0 = J_2 \frac{d^2\theta}{dt^2} + f\frac{d\theta}{dt} + K(\theta - \theta_1) \quad \text{or} \quad 0 = J_2\frac{d\omega}{dt} + f\omega + K\int(\omega - \omega_1)\,dt \qquad \text{(ii)}$$

(in terms of angular displacement) (in terms of angular velocity)

Taking the Laplace transform of Eqs. (i) and (ii), we get

$$T(s) = J_1 s^2\theta_1(s) + K\theta_1(s) - K\theta(s)$$

i.e.
$$T(s) = (J_1 s^2 + K)\theta_1(s) - K\theta(s)$$

and
$$0 = J_2 s^2\theta(s) + fs\theta(s) + K\theta(s) - K\theta_1(s)$$
$$= (J_2 s^2 + fs + K)\theta(s) - K\theta_1(s)$$

$\therefore$
$$\theta_1(s) = \frac{(J_2 s^2 + fs + K)\theta(s)}{K}$$

$\therefore$
$$T(s) = (J_1 s^2 + K)\left(\frac{J_2 s^2 + fs + K}{K}\right)\theta(s) - K\theta(s)$$

$$= (J_1 J_2 s^4 + J_1 fs^3 + J_1 Ks^2 + J_2 Ks^2 + Kfs + K^2 - K^2)\frac{\theta(s)}{K}$$

Therefore, the transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{K}{J_1 J_2 s^4 + J_1 fs^3 + K(J_1 + J_2)s^2 + Kfs}$$

Example 2.17 Write the torque equations of the rotational system shown in Figure 2.39. Also find the transfer function $\frac{\theta_1(s)}{T(s)}$. Obtain the analogous electrical circuits based on torque-current and torque-voltage analogies.

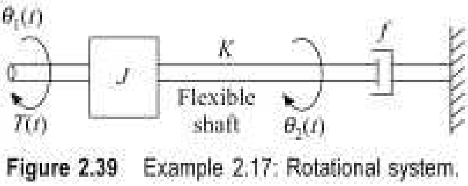


Figure 2.39 Example 2.17: Rotational system.

Solution: The free-body diagram of the given rotational system is shown in Figure 2.40(a). The corresponding mechanical network is shown in Figure 2.40(b).

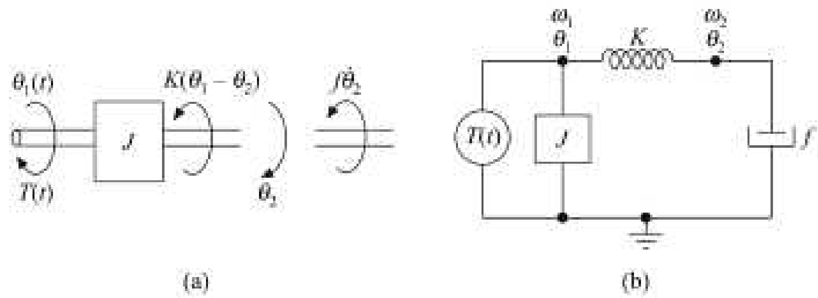


Figure 2.40 Example 2.17: (a) free-body diagram and (b) mechanical network.

The differential equations governing the behavior of the given rotational system are as follows:

$$T(t) = J\frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta_2) \qquad \text{or} \qquad T(t) = J\frac{dv_1}{dt} + K\int (v_1 - v_2)dt \tag{i}$$

and
$$K(\theta_2 - \theta_1) + f\frac{d\theta_2}{dt} = 0 \qquad \text{or} \qquad K\int (v_2 - v_1)\,dt + fv_2 = 0 \tag{ii}$$

Taking the Laplace transform of Eqs. (i) and (ii) and neglecting the initial conditions, we get

$$T(s) = Js^2\theta_1(s) + K[\theta_1(s) - \theta_2(s)]$$

$$= [Js^2 + K]\theta_1(s) - K\theta_2(s) \tag{iii}$$

and
$$K[\theta_2(s) - \theta_1(s)] + fs\theta_2(s) = 0$$

i.e.
$$(fs + K)\theta_2(s) = K\theta_1(s)$$

or
$$\theta_2(s) = \frac{K\theta_1(s)}{fs + K} \tag{iv}$$

Substituting the value of $\theta_2(s)$ from Eq. (iv) in Eq. (iii), we get

$$T(s) = (Js^2 + K)\theta_1(s) - K\left(\frac{K}{fs + K}\right)\theta_1(s)$$

$$= \theta_1(s)\left(\frac{Jfs^3 + JKs^2 + Kfs + K^2 - K^2}{fs + K}\right)$$

$\therefore$
$$\frac{\theta_1(s)}{T(s)} = \frac{fs + K}{s(Jfs^2 + JKs + Kf)}$$

The analogous electrical equations based on force-current analogy are as follows:

$$i(t) = C\frac{d^2\phi_1}{dt^2} + \frac{1}{L}(\phi_1 - \phi_2) \qquad \text{or} \qquad i(t) = C\frac{de_1}{dt} + \frac{1}{L}\int (e_1 - e_2)\,dt$$

and
$$\frac{1}{L}(\phi_2 - \phi_1) + \frac{1}{R}\frac{d\phi_2}{dt} = 0 \qquad \text{or} \qquad \frac{1}{L}\int (e_2 - e_1)\,dt + \frac{1}{R}e_2 = 0$$

The analogous electrical circuit based on force-current analogy is shown in Figure 2.41(a).

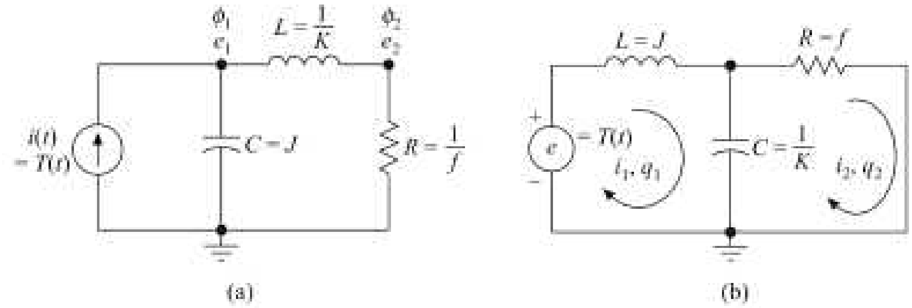


Figure 2.41 Example 2.17: Analogous electrical circuits based on (a) torque-current analogy and (b) torque-voltage analogy.

The electrical equations based on force-voltage analogy are as follows:

$$e(t) = L\frac{d^2q_1}{dt^2} + \frac{1}{C}(q_1 - q_2) \quad \text{i.e.} \quad e(t) = \frac{Ldi_1}{dt} + \frac{1}{C}\int (i_1 - i_2)\, dt$$

and
$$\frac{1}{C}(q_2 - q_1) + R\frac{dq_1}{dt} = 0 \quad \text{i.e.} \quad \frac{1}{C}\int (i_2 - i_1)\, dt + Ri_2 = 0$$

The analogous electrical circuit based on force-voltage analogy is shown in Figure 2.41(b).

Example 2.18 Write the torque equations of the rotational system shown in Figure 2.42. Find the transfer function $\frac{\theta_1(s)}{T(s)}$.

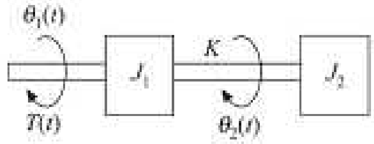


Figure 2.42 Example 2.18: Rotational system.

Solution: The free-body diagram of the rotational system is shown in Figure 2.43(a). The corresponding mechanical network is shown in Figure 2.43(b).

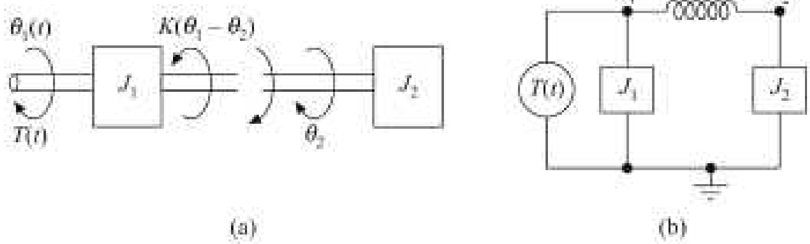


Figure 2.43 Example 2.18: (a) free-body diagram and (b) mechanical network.

The differential equations governing the behaviour of the rotational system are as follows:

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + K(\theta_1 - \theta_2) \quad \text{(i)}$$

and

$$0 = J_2 \frac{d^2\theta_2}{dt^2} + K(\theta_2 - \theta_1) \quad \text{(ii)}$$

Taking the Laplace transform on both sides of Eqs. (i) and (ii), we get

$$T(s) = J_1 s^2 \theta_1(s) + K[\theta_1(s) - \theta_2(s)] = (J_1 s^2 + K)\,\theta_1(s) - K\theta_2(s) \quad \text{(iii)}$$

and

$$0 = J_2 s^2 \theta_2(s) + K[\theta_2(s) - \theta_1(s)]$$

i.e.

$$(J_2 s^2 + K)\,\theta_2(s) = K\theta_1(s)$$

$\therefore$

$$\theta_2(s) = \frac{K}{J_2 s^2 + K}\theta_1(s) \quad \text{(iv)}$$

Substituting the value of $\theta_2(s)$ from Eq. (iv) in Eq. (iii), we get

$$T(s) = (J_1 s^2 + K)\,\theta_1(s) - \frac{K \cdot K\theta_1(s)}{J_2 s^2 + K}$$

Therefore, the transfer function of the system is

$$\frac{\theta_1(s)}{T(s)} = \frac{J_2 s^2 + K}{J_1 J_2 s^4 + (J_2 + J_1)Ks^2} = \frac{J_2 s^2 + K}{s^2[J_1 J_2 s^2 + (J_2 + J_1)K]}$$

Example 2.19 Write the torque equations of the rotational system shown in Figure 2.44. Find the transfer function $\dfrac{\theta_1(s)}{T(s)}$.

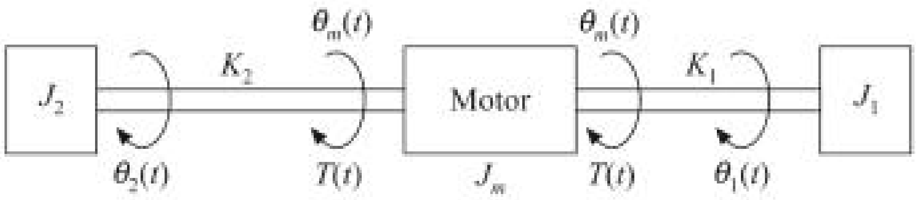


Figure 2.44 Example 2.19: Rotational system.

Solution: The free-body diagram of the given rotational system is shown in Figure 2.45(a). The corresponding mechanical network is shown in Figure 2.45(b).

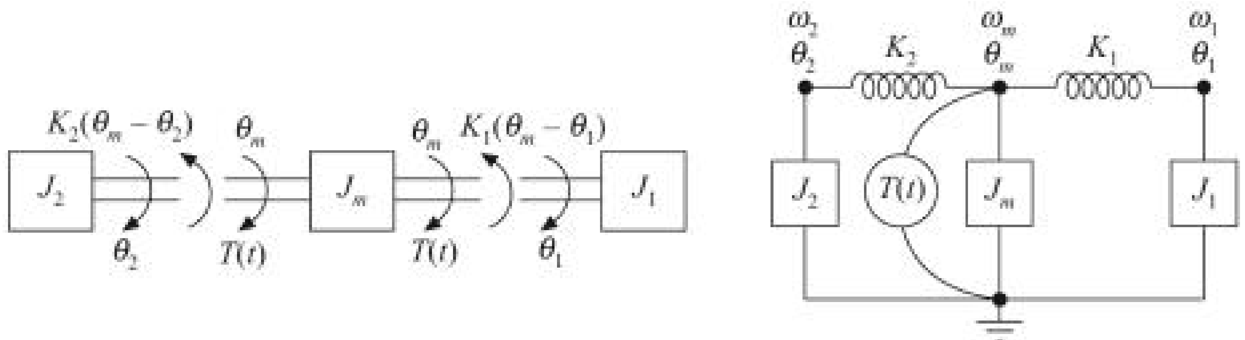


Figure 2.45 Example 2.19: (a) free-body diagram and (b) mechanical network.

The differential equations governing the behaviour of the rotational system are as follows:

$$T(t) = J_m \frac{d^2\theta_m}{dt^2} + K_2(\theta_m - \theta_2) + K_1(\theta_m - \theta_1) \quad \text{(i)}$$

$$J_2 \frac{d^2\theta_2}{dt^2} + K_2(\theta_2 - \theta_m) = 0 \quad \text{(ii)}$$

$$J_1 \frac{d^2\theta_1}{dt^2} + K_1(\theta_1 - \theta_m) = 0 \quad \text{(iii)}$$

Taking the Laplace transform of Eqs. (i), (ii) and (iii) and neglecting the initial conditions,

$$T(s) = J_m s^2\theta_m(s) + K_2[\theta_m(s) - \theta_2(s)] + K_1[\theta_m(s) - \theta_1(s)]$$

i.e.
$$T(s) = [J_m s^2 + (K_2 + K_1)]\ \theta_m(s) - K_2\theta_2(s) - K_1\theta_1(s) \quad \text{(iv)}$$

$$J_2 s^2\theta_2(s) + K_2[\theta_2(s) - \theta_m(s)] = 0$$

i.e.
$$[J_2 s^2 + K_2]\ \theta_2(s) = K_2\theta_m(s) \quad \text{(v)}$$

$$J_1 s^2\theta_1(s) + K_1[\theta_1(s) - \theta_m(s)] = 0$$

i.e.
$$[J_1 s^2 + K_1]\ \theta_1(s) = K_1\theta_m(s) \quad \text{(vi)}$$

$$\therefore \quad \theta_m(s) = \frac{J_2 s^2 + K_2}{K_2}\ \theta_2(s) = \frac{J_1 s^2 + K_1}{K_1}\ \theta_1(s) \quad \text{[from Eqs. (v) and (vi)]}$$

$$\therefore \quad \theta_2(s) = \frac{J_1 s^2 + K_1}{J_2 s^2 + K_2} \times \frac{K_2}{K_1}\ \theta_1(s)$$

Putting the values of $\theta_m(s)$ and $\theta_2(s)$ in Eq. (iv), we get

$$T(s) = [J_m s^2 + (K_2 + K_1)] \left(\frac{J_1 s^2 + K_1}{K_1}\right)\theta_1(s) - K_2\left(\frac{J_1 s^2 + K_1}{J_2 s^2 + K_2}\right)\frac{K_2}{K_1}\theta_1(s) - K_1\theta_1(s)$$

$$= \frac{[J_m J_1 s^4 + J_1(K_2+K_1)s^2 + J_m K_1 s^2 + K_1(K_2+K_1)](J_2 s^2 + K_2)\,\theta_1(s) - [K_2^2 J_1 s^2 + K_2^2 K_1 + K_1^2 J_2 s^2 + K_1^2 K_2]\,\theta_1(s)}{(J_2 s^2 + K_2)K_1}$$

Therefore, the transfer function is

$$\frac{\theta_1(s)}{T(s)} = \frac{(J_2 s^2 + K_2)K_1}{\begin{array}{l} J_1 J_2 J_m s^6 + (K_2 J_m J_1 + J_2 J_1 K_2 + J_2 J_1 K_1 + J_2 J_m K_1)s^4 \\ +(J_2 K_2 K_1 + J_2 K_1^2 + J_1 K_2^2 + J_1 K_1 K_2 + J_m K_1 K_2 - J_1 K_2^2 - J_2 K_1^2)s^2 \\ +K_1^2 K_2 + K_2^2 K_1 - K_1^2 K_2 - K_2^2 K_1 \end{array}}$$

i.e.
$$\frac{\theta_1(s)}{T(s)} = \frac{(J_2 s^2 + K_2)K_1}{\begin{array}{l} J_1 J_2 J_m s^6 + (K_2 J_m J_1 + J_2 J_1 K_2 + J_2 J_1 K_1 + J_2 J_m K_1)s^4 \\ +(J_2 K_2 K_1 + J_1 K_1 K_2 + J_m K_1 K_2)s^2 \end{array}}$$

Example 2.20 Write the torque equations of the rotational system shown in Figure 2.46. Draw the analogous electrical networks based on (a) torque-current and (b) torque-voltage analogies.

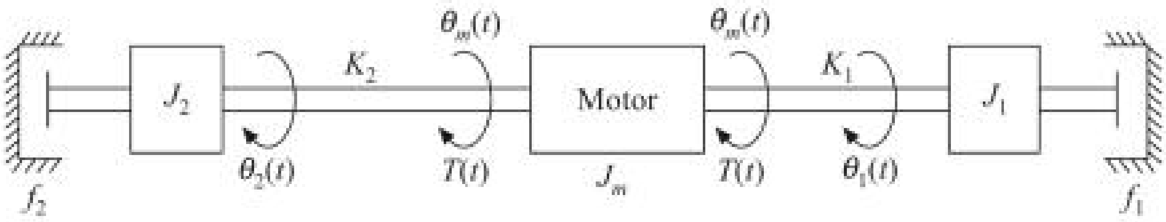


Figure 2.46 Example 2.20: Rotational system.

Solution: The free-body diagram and the mechanical network corresponding to the given rotational system are shown in Figures 2.47(a) and 2.47(b).

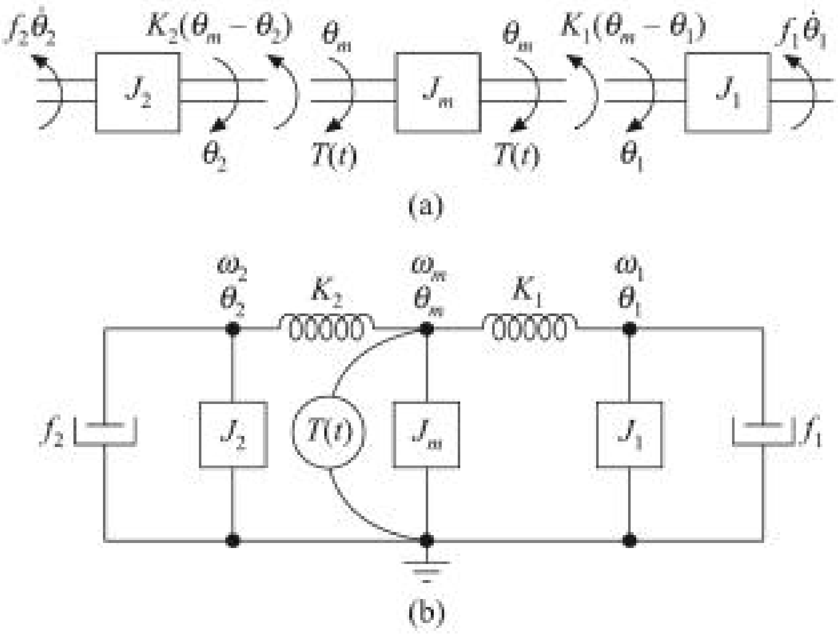


Figure 2.47 Example 2.20: (a) free-body diagram and (b) mechanical network.

The differential equations for the rotational system are as follows:

$$T(t) = J_m \frac{d^2\theta_m}{dt^2} + K_2(\theta_m - \theta_2) + K_1(\theta_m - \theta_1)$$

or
$$T(t) = J_m \frac{d\omega_m}{dt} + K_2 \int (\omega_m - \omega_2)\, dt + K_1 \int (\omega_m - \omega_1)\, dt$$

$$J_1 \frac{d^2\theta_1}{dt^2} + f_1 \frac{d\theta_1}{dt} + K_1(\theta_1 - \theta_m) = 0$$

or
$$J_1 \frac{d\omega_1}{dt} + f_1\omega_1 + K_1 \int (\omega_1 - \omega_m) dt = 0$$

$$J_2 \frac{d^2\theta_2}{dt^2} + f_2 \frac{d\theta_2}{dt} + K_2(\theta_2 - \theta_m) = 0$$

or
$$J_2 \frac{d\omega_2}{dt} + f_2\omega_2 + K_2 \int (\omega_2 - \omega_m)\, dt = 0$$

The analogous electrical network based on torque-current analogy is shown in Figure 2.48. The analogous electrical network based on torque-voltage analogy is shown in Figure 2.49.

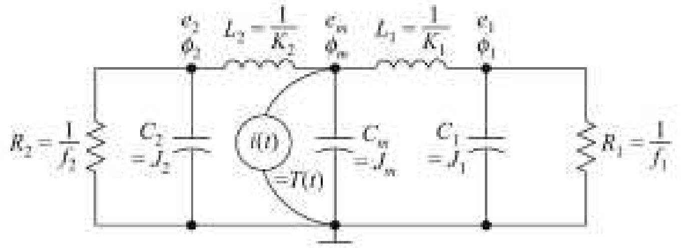


Figure 2.48 Example 2.20: Analogous electrical network based on torque-current analogy.

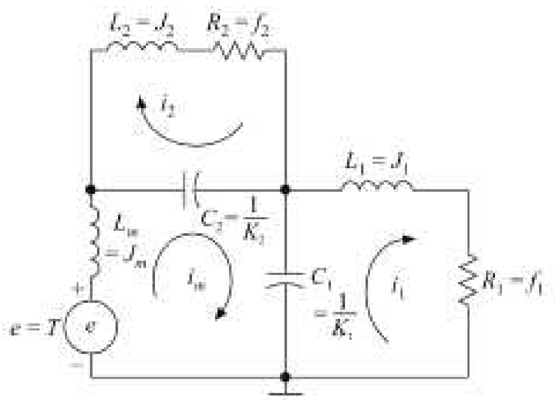


Figure 2.49 Example 2.20: Analogous electrical network based on torque-voltage analogy.

Example 2.21 Write the differential equations governing the mechanical rotational system shown in Figure 2.50 and determine the transfer function $\theta(s)/T(s)$.

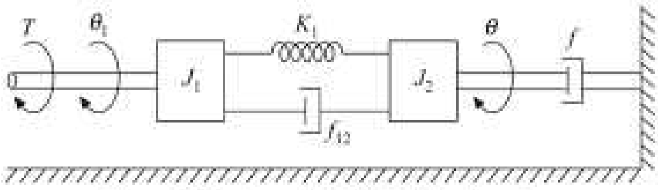


Figure 2.50 Example 2.21: Rotational system.

Solution: The free-body diagram and the mechanical network corresponding to the given rotational system are shown in Figures 2.51 and 2.52 respectively.

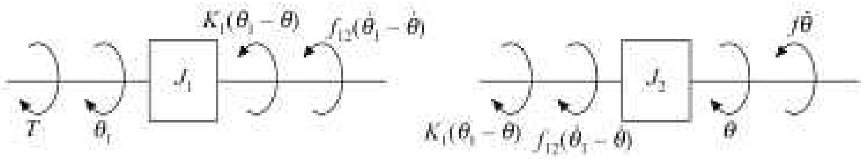


Figure 2.51 Example 2.21: Free-body diagram.

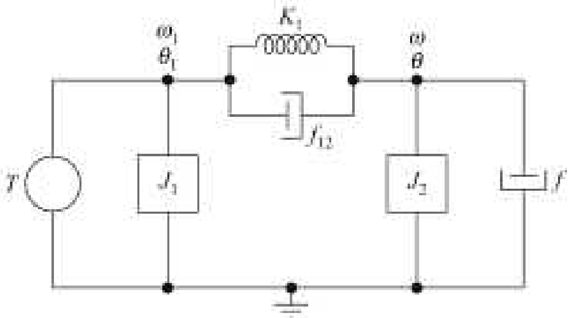


Figure 2.52 Example 2.21: Mechanical network.

The differential equations for the rotational system based on Newton's law of motion (which states that the algebraic sum of torques acting on a rigid body is equal to the product of the moment of inertia and angular acceleration of the body) are as follows:

$$T - K_1(\theta_1 - \theta) - f_{12}\left(\frac{d\theta_1}{dt} - \frac{d\theta}{dt}\right) = J_1 \frac{d^2\theta_1}{dt^2} \tag{i}$$

or

$$T = J_1 \frac{d\omega_1}{dt} + f_{12}(\omega_1 - \omega) + K_1 \int (\omega_1 - \omega)\, dt = 0$$

$$K_1(\theta_1 - \theta) + f_{12}\left(\frac{d\theta_1}{dt} - \frac{d\theta}{dt}\right) - f\frac{d\theta}{dt} = J_2 \frac{d^2\theta}{dt} \tag{ii}$$

or

$$0 = J_2 \frac{d\omega}{dt} + f\omega + f_{12}(\omega - \omega_1) + K_1 \int (\omega - \omega_1)\, dt$$

Taking the Laplace transform of Eqs. (i) and (ii) with zero initial conditions,

$$T(s) = J_1 s^2 \theta_1(s) + f_{12}[s\theta_1(s) - s\theta(s)] + K_1[\theta_1(s) - \theta(s)]$$

i.e.
$$T(s) = (J_1 s^2 + f_{12}s + K_1)\,\theta_1(s) - (f_{12}s + K_1)\theta(s) \qquad \text{(iii)}$$

$$0 = J_2 s^2 \theta(s) + fs\theta(s) + f_{12}[s\theta(s) - s\theta_1(s)] + K_1[\theta(s) - \theta_1(s)]$$

i.e.
$$(J_2 s^2 + fs + f_{12}s + K_1)\,\theta(s) - (f_{12}s + K_1)\,\theta_1(s) = 0$$

$\therefore$
$$\theta_1(s) = \frac{(J_2 s^2 + fs + f_{12}s + K_1)\,\theta(s)}{f_{12}s + K_1} \qquad \text{(iv)}$$

Substituting the value of $\theta_1(s)$ from Eq. (iv) in Eq. (iii), we get

$$T(s) = (J_1 s^2 + f_{12}s + K_1)\,\frac{[J_2 s^2 + (f + f_{12})s + K_1]\,\theta(s)}{f_{12}s + K_1} - (f_{12}s + K_1)\,\theta(s)$$

i.e.
$$T(s) = \left[\frac{J_1 J_2 s^4 + (J_2 f_{12} + J_1 f + J_1 f_{12})s^3 + (J_1 K_1 + J_2 K_1 + ff_{12} + f_{12}^{\ 2})s^2 + (2K_1 f_{12} + K_1 f)s + K_1^{\ 2} - f_{12}^{\ 2}s^2 - K_1^{\ 2} - 2f_{12}sK_1}{f_{12}s + K_1}\right]\theta(s)$$

i.e.
$$T(s) = \left[\frac{J_1 J_2 s^4 + (J_2 f_{12} + J_1 f + J_1 f_{12})s^3 + (J_1 K_1 + J_2 K_1 + ff_{12})\,s^2 + K_1 fs}{f_{12}s + K_1}\right]\theta(s)$$

Therefore, the transfer function is

$$\frac{\theta(s)}{T(s)} = \frac{f_{12}s + K_1}{s[J_1 J_2 s^3 + (J_2 f_{12} + J_1 f + J_1 f_{12})s^2 + (J_1 K_1 + J_2 K_1 + ff_{12})s + K_1 f]}$$

2.7 SERVOMOTORS

The control systems which are used to control the position or time derivatives of position, i.e. velocity and acceleration are called *servomechanisms*. The motors which are used in automatic control systems are called *servomotors*. The servomotors are used to convert an electrical signal (control voltage) applied to them into an angular displacement of the shaft. Depending on the construction, they can operate either in a continuous duty or step duty.

A variety of servomotors are available for control system applications. The suitability of a motor for a particular application depends on the characteristics of the system, the purpose of the system and its operating conditions.

In general, a servomotor should have the following features:

- Linear relationship between speed and electric control signal
- Steady-state stability

- Wide range of speed control
- Linearity of mechanical characteristics throughout the entire speed range
- Low mechanical and electrical inertia
- Fast response

Servomotors are broadly classified as dc servomotors and ac servomotors depending on the power supply required to run the motor. Eventhough dc motors are costlier than ac motors, they have linear characteristics and so it is easier to control. They are generally used for large power applications such as in machine tools and robotics.

The advantages of dc servomotors are as follows:

1. Higher output than from an ac motor of the same size.
2. Easy achievement of linear characteristics.
3. Easier speed control from zero speed to full speed in both the directions.
4. High torque to inertia ratio that gives them quick response to control signals.
5. The dc servomotors have light weight, low inertia and low inductance armature that can respond quickly to commands for a change in position or speed.
6. Low electrical and mechanical time constants.
7. The dc motors are capable of delivering over 3 times their rated torque for a short time compared to the 2 to 2.5 times the rated torque developed by the ac motors.

The advantages of ac servomotors are lower cost, higher efficiency and less maintenance since the brushes and commutator are not there. The disadvantages of ac motors are their characteristics are quite nonlinear and these motors are more difficult to control especially for positioning applications.

The ac motors are best suited for low power applications, such as instrument servo (e.g., control of pen in *x-y* recorders) and computer-related equipment (e.g., disc drives, tape drives, printers, etc.). The three-phase induction motors with pulse width modulation power amplifier are currently gaining popularity in high power control applications.

2.7.1 DC Servomotors

The dc servomotors are broadly classified into (a) sliding contact motors with commutator and brushes and (b) brushless or contact less motors with SCR/transistor commutator.

The sliding contact motors may be classified into (a) permanent magnet motors and (b) electromagnetic field motors.

The permanent magnet motors may be (a) cylindrical armature motors, or (b) disc armature motors, or (c) moving coil motors.

The electromagnetic field motors may be (a) armature-controlled motors, or (b) field-controlled motors, or (c) series motors, or (d) split field motors.

Permanent magnet dc motors: In this type of motors, the armature is placed in rotor and the field winding is replaced by permanent magnet poles fixed to the stator to produce the required magnetic field. Permanent magnet motors are economical for power ratings up to a few kilowatts.

The following are some of the advantages of permanent magnet motors:

1. A simpler and more reliable motor because the field power supply is not required.
2. Higher operating efficiency as the motor has no field losses.
3. Field flux is less affected by temperature rise.
4. Higher torque/inertia ratio.
5. Speed is nearly directly proportional to armature voltage at a given load torque.
6. A more linear torque/speed curve.
7. Higher output power at the same dimensions and temperature limitations.

The disadvantages of permanent magnet motors are: the magnets deteriorate with time and demagnetized with large current transients. These drawbacks are eliminated by high-grade magnetic materials such as ceramic magnets and rear earth magnets. But the cost of these materials is very high.

Electromagnetic field dc motors: Electromagnetic motors are economical for higher power ratings generally above 1 kW. This type of servomotor is similar to a conventional dc motor constructionally but has the following special features:

1. The number of slots and commutator segments is large to improve commutation.
2. Compoles and compensating windings are provided to eliminate sparking.
3. The diameter to length ratio is kept low to reduce inertia.
4. Oversize shafts are employed to withstand the high torque stress.
5. Eddy currents are reduced by complete elimination of the magnetic circuit and by using low loss steel.

In this type of motor, the torque and speed may be controlled by varying the armature current and/or the field current. Generally, one of these is varied to control the torque and the other is held constant. In armature-controlled mode of operation, the field current is held constant and the armature current is varied to control the torque. In the field-controlled mode, the armature current is maintained constant and field current is varied to control the torque.

In servo applications, dc motors are required to produce rapid accelerations from standstill. Therefore, the physical requirements of such a motor are low inertia and high starting torque. Low inertia is attained with reduced armature diameter with a consequent increase in armature length such that the desired power output is achieved. Thus, except for minor differences in constructional features, a dc servomotor is essentially an ordinary dc motor.

In control systems, the dc motors are used mainly in two different control modes: armature control mode with fixed field current and field control mode with fixed armature current.

Armature-controlled dc servomotor: An armature-controlled dc servomotor is a dc shunt motor designed to satisfy the requirement of a servomotor. If the field current is constant, then speed is directly proportional to armature voltage, and torque is directly proportional to armature current. Hence the torque and speed can be controlled by armature voltage. Reversible operation is possible by reversing the armature voltage.

In small motors, the armature voltage is controlled by a variable resistance. But in large motors in order to reduce power loss, armature voltage is controlled by thyristors.

Figure 2.53 shows an armature-controlled dc motor.

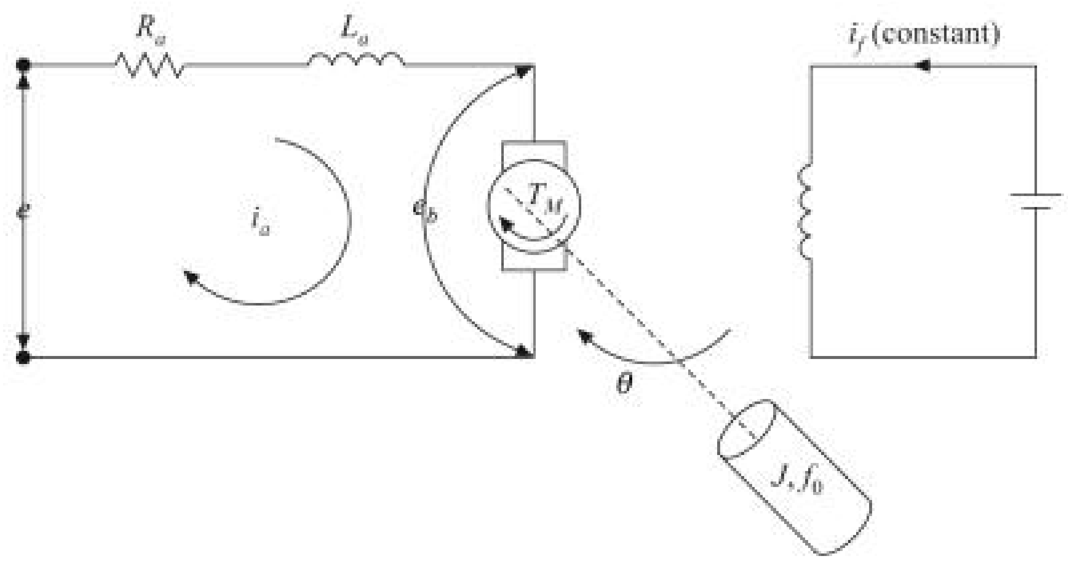


Figure 2.53 Armature-controlled dc motor.

In this system,

R_a = resistance of armature winding (in ohms)
L_a = inductance of armature winding (in henrys)
i_a = armature current (in amperes)
i_f = field current (in amperes)
e = applied armature voltage (in volts)
e_b = back emf (in volts)
T_M = torque developed by motor (in N-m)
θ = angular displacement of motor shaft (in radians)
J = equivalent moment of inertia of motor and load referred to motor shaft (in kg-m^2)
f_0 = equivalent viscous friction coefficient of motor and load referred to motor shaft (in N-m/rad/s)

In servo applications, the dc motors are generally used in the linear range of the magnetization curve. Therefore, the air gap flux ϕ is proportional to the field current, i.e.

$$\phi \propto i_f$$

or

$$\phi = K_f \, i_f \tag{2.8}$$

where K_f is a constant.

The torque T_M developed by the motor is proportional to the product of the air gap flux ϕ and the armature current i_a, i.e.

$$T_M \propto \phi i_a$$

i.e.

$$T_M \propto K_f \, i_f \, i_a$$

or

$$T_M = K_1 \, K_f \, i_f \, i_a \tag{2.9}$$

where K_1 is a constant.

In the armature-controlled dc motor, field current is kept constant, so the equation for T_M can be written as

$$T_M = K_T\, i_a \tag{2.10}$$

where K_T is known as the motor torque constant.

The motor back emf is proportional to speed, i.e.

$$e_b \propto \frac{d\theta}{dt}$$

or

$$e_b = K_b \frac{d\theta}{dt} \tag{2.11}$$

where K_b is the back emf constant.

The differential equation of the armature circuit is

$$e = L_a \frac{di_a}{dt} + R_a i_a + e_b \tag{2.12}$$

The torque equation is

$$T_M = K_T i_a = J\frac{d^2\theta}{dt^2} + f_0 \frac{d\theta}{dt} \tag{2.13}$$

Taking the Laplace transform of Eqs. (2.11)–(2.13),

$$E_b(s) = K_b s\theta(s) \tag{2.14}$$

$$E(s) = L_a s I_a(s) + R_a I_a(s) + E_b(s) \tag{2.15}$$

$$I_a(s)(L_a s + R_a) = E(s) - E_b(s)$$

i.e.

$$I_a(s) = \frac{E(s) - E_b(s)}{(L_a s + R_a)} \tag{2.16}$$

$$Js^2\theta(s) + f_0 s\theta(s) = T_M(s) = K_T I_a(s)$$

i.e.

$$\theta(s)(Js^2 + f_0 s) = K_T I_a(s)$$

i.e.

$$s\theta(s)[Js + f_0] = K_T \frac{[E(s) - E_b(s)]}{(L_a s + R_a)} = K_T \frac{[E(s) - K_b s\theta(s)]}{(L_a s + R_a)}$$

[Substituting the values of $I_a(s)$ and $E_b(s)$ from Eqs. (2.16) and (2.14)]

i.e.

$$(L_a s + R_a)s(Js + f_0)\theta(s) = K_T E(s) - K_T K_b s\theta(s)$$

i.e.

$$s[(L_a s + R_a)(Js + f_0) + K_T K_b]\theta(s) = K_T E(s)$$

Therefore, the transfer function is

$$\frac{\theta(s)}{E(s)} = \frac{K_T}{s[(L_a s + R_a)(Js + f_0) + K_T K_b]}$$

The block diagram of the dc motor can be obtained as follows. The block diagram representation of the equation

$$I_a(s) = \frac{E(s) - E_b(s)}{(L_a s + R_a)}$$

is shown in Figure 2.54(a), where the circular block representing the differencing action is known as the *summing point*.

The block diagram representation of the equation.

$$\theta(s) = I_a(s)\frac{K_T}{s(Js + f_0)}$$

is shown in Figure 2.54(b). Here the signal is taken off from a take-off point and fed to the feedback block. The block diagram representation of the equation

$$E_b(s) = K_b s\theta(s)$$

is shown in Figure 2.54 (c).

Figure 2.54(d) shows the complete block diagram of the system under consideration obtained by connecting the block diagrams shown in Figure 2.54(a), Figure 2.54(b) and Figure 2.54(c).

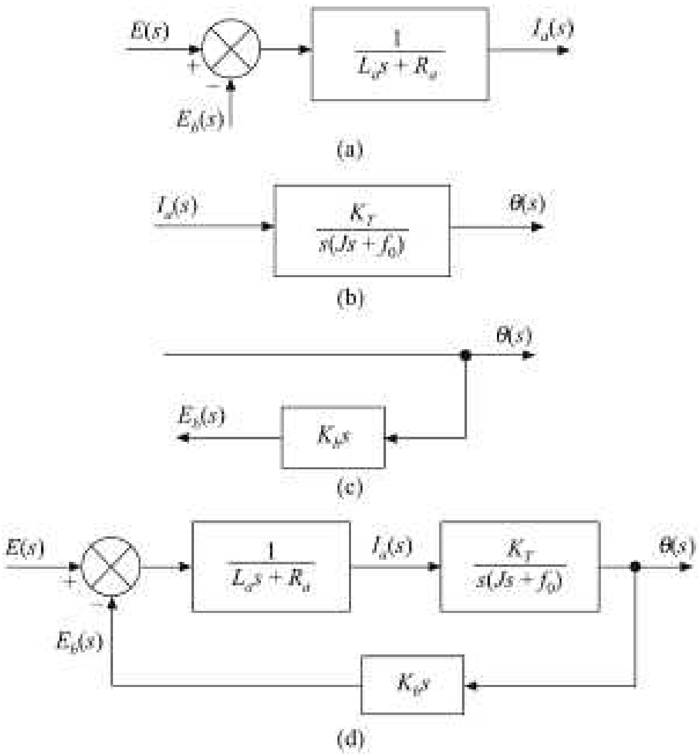


Figure 2.54 Block diagram of armature-controlled dc motor.

The block diagram of the system under consideration can be directly obtained from the physical system of Figure 2.53 by using the mechanical networks derived already.

The voltage applied to the armature circuit is $E(s)$ which is opposed by the back emf $E_b(s)$. The net voltage $E(s) - E_b(s)$ acts on a linear circuit comprised of resistance and inductance in series, having the transfer function $1/(L_a s + R_a)$. The result is an armature current $I_a(s)$. Since the field is fixed, the torque rotates the load at a speed $\theta(s)$ against the moment of inertia J and the viscous friction with a coefficient f_0. The transfer function is $1/(Js + f_0)$. The back emf signal $E_b(s) = K_b\, s\theta(s)$ is taken off from the shaft speed and fed back negatively to the summing point. The angle signal $\theta(s)$ is obtained by integrating (i.e., by multiplying by $1/s$) the speed $s\theta(s)$. This results in the block diagram of Figure 2.55, which is equivalent to that of Figure 2.54.

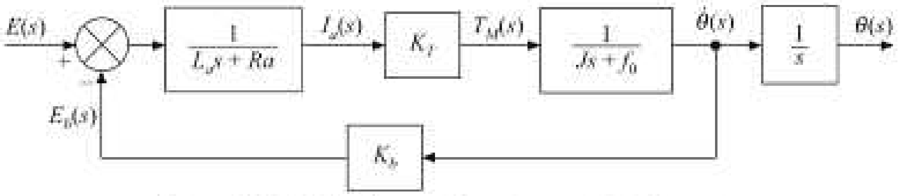


Figure 2.55 Block diagram of armature-controlled dc motor.

The armature circuit inductance L_a is usually negligible. Therefore, the transfer function of the armature-controlled motor simplifies to

$$\frac{\theta(s)}{E(s)} = \frac{\dfrac{K_T}{R_a}}{Js^2 + s\left(f_0 + \dfrac{K_T K_b}{R_a}\right)} \tag{2.17}$$

The term $\left(f_0 + \dfrac{K_T K_b}{R_a}\right)$ of Eq. (2.17) indicates that the back emf of the motor effectively increases the viscous friction of the system. Let $f = f_0 + \dfrac{K_T K_b}{R_a}$ be the effective viscous friction coefficient. Then, Eq. (2.17) becomes

$$\frac{\theta(s)}{E(s)} = \frac{\dfrac{K_T}{R_a}}{s(Js + f)} = \frac{\dfrac{K_T}{R_a f}}{s[(Js/f) + 1]} \tag{2.18}$$

Equation (2.18) may be written as

$$\frac{\theta(s)}{E(s)} = \frac{K_m}{s(\tau_m s + 1)}$$

where $K_m = \dfrac{K_T}{R_a f}$ = motor gain constant

$T_m = \dfrac{J}{f}$ = motor time constant

Field-controlled dc servomotor: A field-controlled dc servomotor, is a dc shunt motor designed to satisfy the requirement of a servomotor. In this motor, the armature is supplied with a constant current or voltage. When armature voltage is constant, the torque is directly proportional to the field flux. Since the field current is proportional to flux, the torque of the motor is controlled by controlling the field current. The response of a field-controlled motor is however slowed by field inductance.

A field controlled dc motor is shown in Figure 2.56(a).

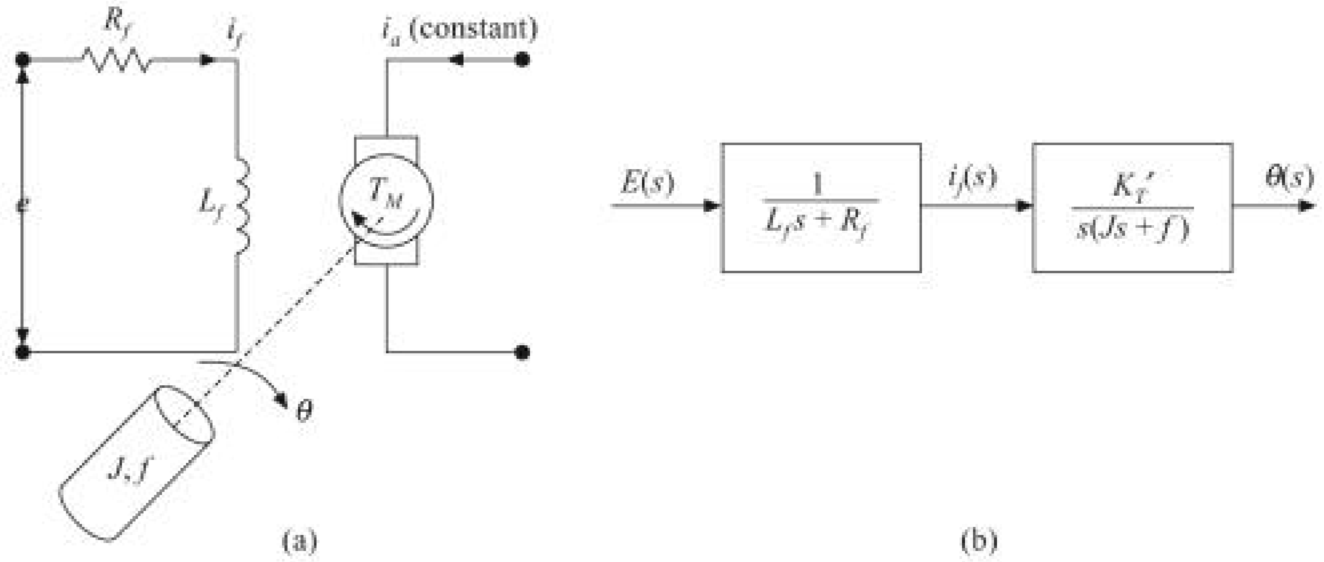


Figure 2.56 (a) Field controlled dc motor and (b) block diagram of field controlled dc motor.

In this system,

R_f = field winding resistance (in ohms)
L_f = field winding inductance (in henrys)
e = field control voltage (in volts)
i_f = field current (in amperes)
T_M = torque developed by motor (in N-m)
J = equivalent moment of inertia of motor and load referred to motor shaft (in kg-m^2)
f = equivalent viscous friction coefficient of motor and load referred to motor shaft (in N-m/rad/s)
θ = angular displacement of motor shaft (in radians)

In the field-controlled motor, the armature current is fed from a constant current source. Since the motor is operating in the linear region of the magnetization curve, the flux is proportional to the field current, i.e.

$$\phi \propto i_f$$

or

$$\phi = K_f \, i_f$$

where K_f is a constant.

The torque developed by the motor

$$T_M \propto \phi\, i_a$$

$\therefore$ $$T_M \propto K_f\, i_f\, i_a$$

or $$T_M = K_T\, K_f\, i_a\, i_f = K_T' i_f$$

where K_T' is a constant.

The equation for the field circuit is

$$L_f \frac{di_f}{dt} + R_f i_f = e \tag{2.19}$$

The torque equation is

$$J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt} = T_M = K_T' i_f \tag{2.20}$$

Taking the Laplace transform of Eqs. (2.19) and (2.20), assuming zero initial conditions, we get

$$(L_f s + R_f) I_f(s) = E(s)$$

i.e. $$I_f(s) = \frac{E(s)}{L_f s + R_f} \tag{2.21}$$

$$(Js^2 + fs)\theta(s) = T_M(s) = K_T' I_f(s) = \frac{K_T' E(s)}{(L_f s + R_f)}$$

[Substituting the value of $I_f(s)$ from Eq. (2.21)]

Therefore, the transfer function is

$$\frac{\theta(s)}{E(s)} = \frac{K_T'}{s(L_f s + R_f)(Js + f)} = \frac{\dfrac{K_T'}{R_f f}}{s\left(\dfrac{L_f s}{R_f} + 1\right)\left(\dfrac{Js}{f} + 1\right)} = \frac{K_m}{s(\tau_f s + 1)(\tau_m s + 1)}$$

where $K_m = \dfrac{K_T'}{R_f f}$ = motor gain constant

$\tau_f = \dfrac{L_f}{R_f}$ = time constant of field circuit

$\tau_m = \dfrac{J}{f}$ = mechanical time constant

The block diagram of the field controlled dc motor is shown in Figure 2.56(b).

Comparison of armature-controlled and field-controlled modes: For small size motors, field control is advantageous because only a low power servoamplifier is required, while the armature current which is not large can be supplied from an inexpensive current amplifier. For large size motors, it is on the whole cheaper to use armature control scheme. Further, in an armature-controlled motor, back emf contributes additional damping over and above that provided by load friction.

2.7.2 AC Servomotors

An ac servomotor is basically a two-phase induction motor except for certain special design features. A two-phase induction motor consists of two stator windings oriented 90°.

A two-phase servomotor differs in the following two ways from a normal induction motor.

1. The rotor of the servomotor is built with high resistance so that its X/R (inductive reactance/resistance) ratio is small which results in linear speed-torque characteristic, as shown by curve b in Figure 2.57. But the conventional induction motors have large X/R ratio which results in high efficiency but in highly nonlinear torque speed characteristics as shown by curve a in Figure 2.57. Also because of the positive slope for part of the characteristic, the system using such a motor becomes unstable.
2. The excitation voltage applied to two stator windings should have a phase difference of 90°.

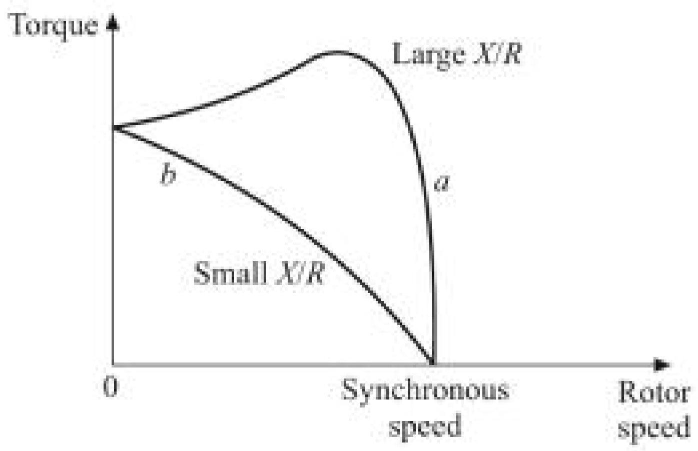


Figure 2.57 Torque-speed characteristics of induction motor.

Construction of ac servomotor: An ac servomotor is basically a two-phase induction motor with some special design features. The stator consists of two pole pairs (A-B and C-D) mounted on the inner periphery of the stator such that their axes are at an angle of 90° in space. Each pole pair carries a winding. One winding is called reference winding and the other is called control winding. The exciting current in the winding should have a phase displacement of 90°. The supply used to drive the motor is single-phase and so a phase advancing capacitor is connected to one of the phases to produce a phase difference of 90°. The constructional features of an ac servomotor are shown in Figure 2.58.

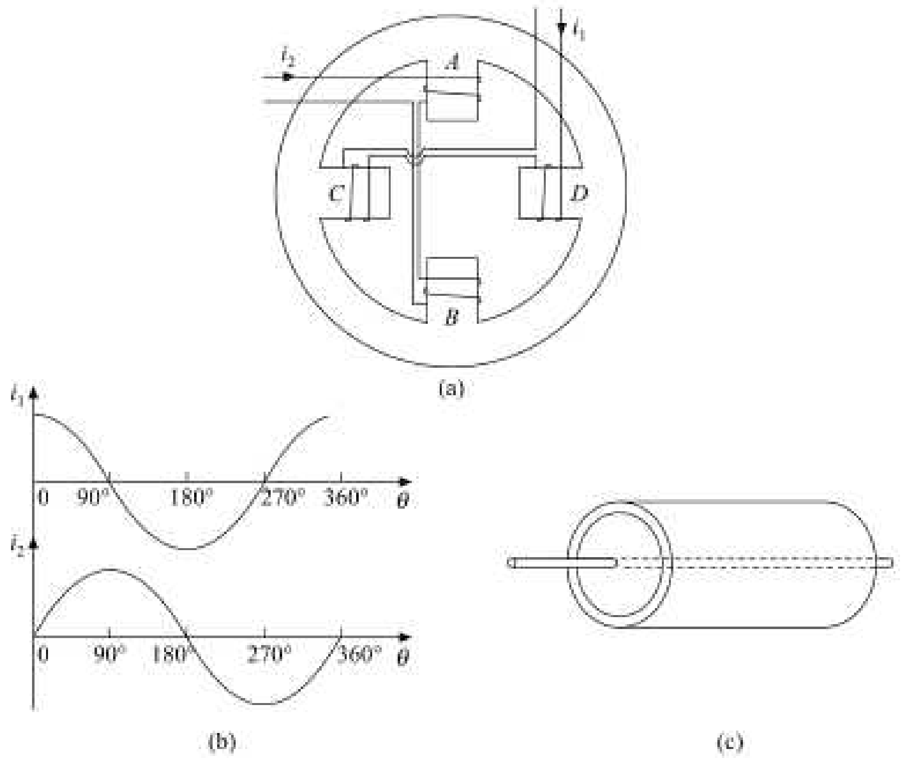


Figure 2.58 Simplified constructional features of a two-phase ac servo motor: (a) stator, (b) exciting currents and (c) rotor.

The rotor construction is usually squirrel cage or drag-cup type. The squirrel cage rotor is made of laminations. The rotor bars are placed in the slots and short-circuited at both ends by end rings. The diameter of the rotor is kept small in order to reduce inertia and to obtain good accelerating characteristics.

The drag-cup construction is employed for very low inertia applications. In this type of construction, the rotor will be in the form of a hollow cylinder made of aluminum. The aluminum cylinder itself acts as short-circuited rotor conductors (electrically both types of rotors are identical).

Working of a two-phase induction motor: Figure 2.59 shows a schematic diagram for balanced operation of the two-phase induction motor. The stator windings are excited by voltages of equal rms magnitude and 90° phase difference. This results in exciting currents i_1 and i_2 that are phase displaced by 90° and have equal rms value. So their respective fields will be 90° apart in both time and space resulting in a magnetic field of constant magnitude rotating at synchronous speed. The direction of rotation depends upon phase relationship of the two currents (voltages). As the field sweeps over the rotor, voltages are induced in it producing current in the short-circuited rotor. The rotating magnetic field interacts with these currents producing a torque on the rotor in the direction of field of rotation and so the rotor starts moving in the same direction as that of rotating magnetic field.

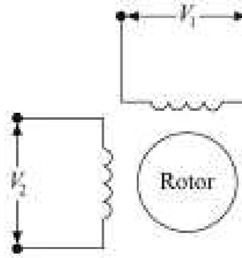


Figure 2.59 Schematic diagram of a two-phase induction motor.

Working of an ac servomotor: In servo applications, the voltages applied to the two stator windings are rarely balanced. Figure 2.60. shows a schematic diagram of an ac servomotor as a control system component. As shown in Figure 2.60, one of the phases known as the reference phase is excited by a constant voltage, and the other phase known as the control phase is energized by a voltage which is 90° out of phase with respect to the voltage of the reference phase. The control phase voltage is supplied from a servo amplifier and it has a variable magnitude and polarity (±90° phase angle with respect to the reference phase). The direction of rotation of the motor reverses as the polarity of the control phase signal changes in sign.

The control signals in control systems are usually of low frequency, in the range of 0 to 20 Hz. For production of rotating magnetic field, the control phase voltage must be of the same frequency as the reference phase voltage and in addition the two voltages must be in time quadrature. Hence, the control signal is modified by a carrier whose frequency is the same as that of the reference voltage and then applied to control winding. The ac supply itself is used as carrier signal for modulation process. The 90°-phase difference between the control-phase and the reference-phase voltages is obtained by the insertion of a capacitor in reference winding.

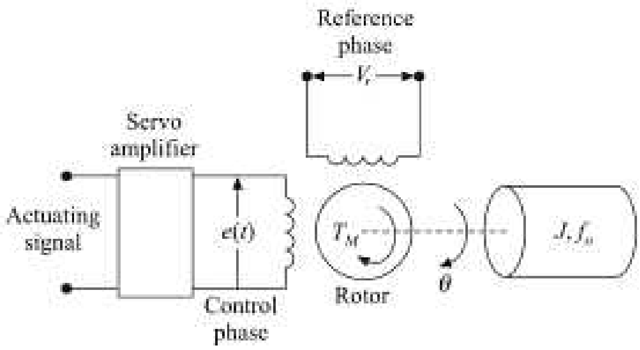


Figure 2.60 Schematic diagram of a two-phase servomotor.

The torque-speed curves of a typical ac servomotor plotted for fixed reference phase voltage and with variable rms control voltage are shown in Figure 2.61. All these curves have a negative slope. Note that the curve for zero control voltage goes through the origin. This means that when the control phase voltage becomes zero, the motor develops a decelerating torque and so the motor stops. The curves show a large torque at zero speed. This is a requirement for a servomotor in order to provide rapid acceleration. The torque-speed curves of ac servomotors are nonlinear except in the low speed region.

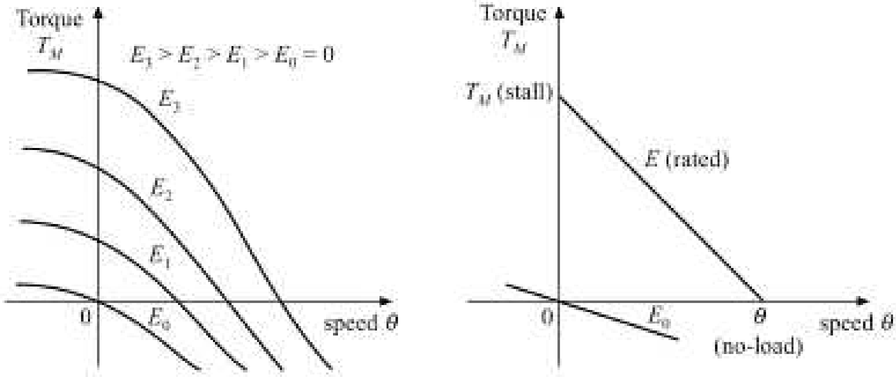


Figure 2.61 Servomotor characteristics.

Transfer function of an ac servomotor: Let

T_M = torque developed by servomotor
θ = angular displacement of rotor
$\omega = d\theta/dt$ = angular speed
T_l = torque required by the load
J = moment of inertia of load and the rotor
K_1 = slope of control-phase voltage versus torque characteristics
K_2 = slope of torque-speed characteristic

With reference to Figure 2.61, we can say that for speeds near zero, all the curves are straight lines parallel to the characteristic at rated input voltage ($e_c = E$) and are equally spaced for equal increments of the input voltage. Under this assumption, the torque developed by the motor is

$$T_M = K_1 e_c - K_2 \frac{d\theta}{dt} \tag{2.22}$$

The load torque is given by

$$T_l = J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt} \tag{2.23}$$

At equilibrium, the motor torque is equal to the load torque. Therefore,

$$\therefore \quad J \frac{d^2\theta}{dt^2} + f \frac{d\theta}{dt} = K_1 e_c - K_2 \frac{d\theta}{dt} \tag{2.24}$$

Taking the Laplace transform on both sides of Eq. (2.24) and neglecting initial conditions, we get

$$Js^2\theta(s) + fs\theta(s) = K_1E_c(s) - K_2s\theta(s)$$

i.e.

$$(Js^2 + fs + K_2s)\ \theta(s) = K_1E_c(s)$$

The transfer function of the system is therefore

$$\frac{\theta(s)}{E_c(s)} = \frac{K_1}{Js^2 + fs + K_2s} = \frac{K_1}{s(f+K_2)\left[\dfrac{Js^2}{s(f+K_2)} + 1\right]}$$

$$= \frac{K_1/(f+K_2)}{s\left(\dfrac{J}{f+K_2}s + 1\right)} = \frac{K_m}{s(\tau_m s + 1)}$$

where $K_m = \dfrac{K_1}{f+K_2}$ = motor gain constant

$\tau_m = \dfrac{J}{f+K_2}$ = motor time constant

2.8 SYNCHROS

A *synchro* is an electromagnetic *transducer* commonly used to convert an angular position of a shaft into an electric signal. It is commercially known as *selsyn* or *autosyn.* It produces an output voltage depending upon the angular position of the rotor. It works on the principle of an induction motor.

2.8.1 Synchro Transmitter

The basic synchro unit is usually called a *synchro transmitter*. Its construction is similar to that of a three-phase alternator. The two major parts of a synchro transmitter are stator and rotor. The stator (stationary member) is of laminated silicon steel and is slotted on the inner periphery to accommodate a balanced three-phase winding which is usually of concentric coil type (three identical coils are placed in the stator with their axes 120° apart) and is Y connected. The rotor is of dumb-bell construction with a single winding and is wound with a concentric coil. A single-phase ac voltage is applied to the rotor winding through slip rings. The constructional features and a schematic diagram of a synchro transmitter are shown in Figures 2.62. and 2.63 respectively.

Let an ac voltage

$$v_r(t) = V_r \sin\ \omega_c(t) \tag{2.25}$$

be applied to the rotor of the synchro transmitter as shown in Figure 2.63. This voltage causes a flow of magnetizing current in the rotor coil which produces a sinusoidally time varying flux directed along its axis and distributed nearly sinusoidally in the air gap along the stator periphery. Because of transformer action, voltages are induced in each of the stator coils. As the air gap flux is sinusoidally distributed, the flux linking stator coil is proportional to the cosine of the angle between the rotor and stator coil axes and so is the voltage induced in each stator coil. The stator coil voltages are of course in time phase with each other. Thus we see that the synchro transmitter acts like a single-phase transformer in which the rotor coil is the primary and the stator coils form the three secondaries.

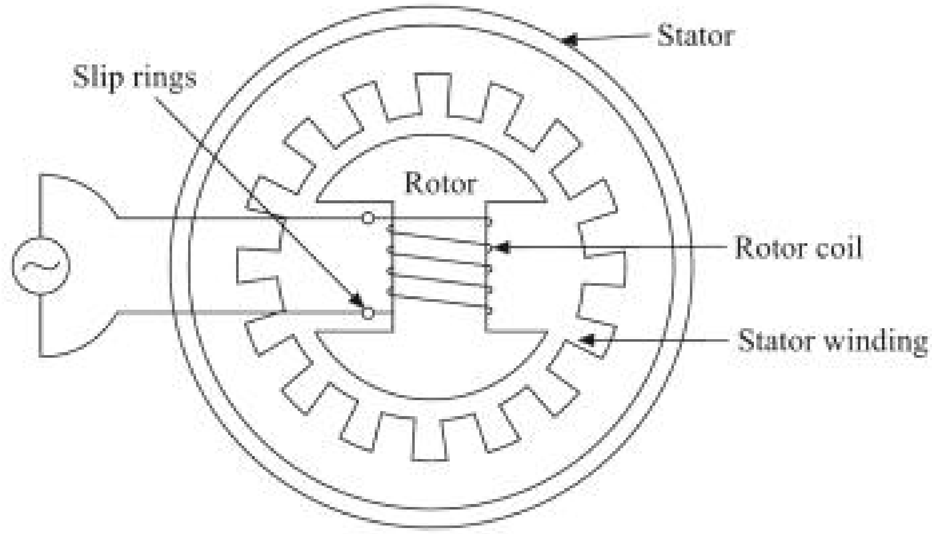


Figure 2.62 Constructional features of a synchro transmitter.

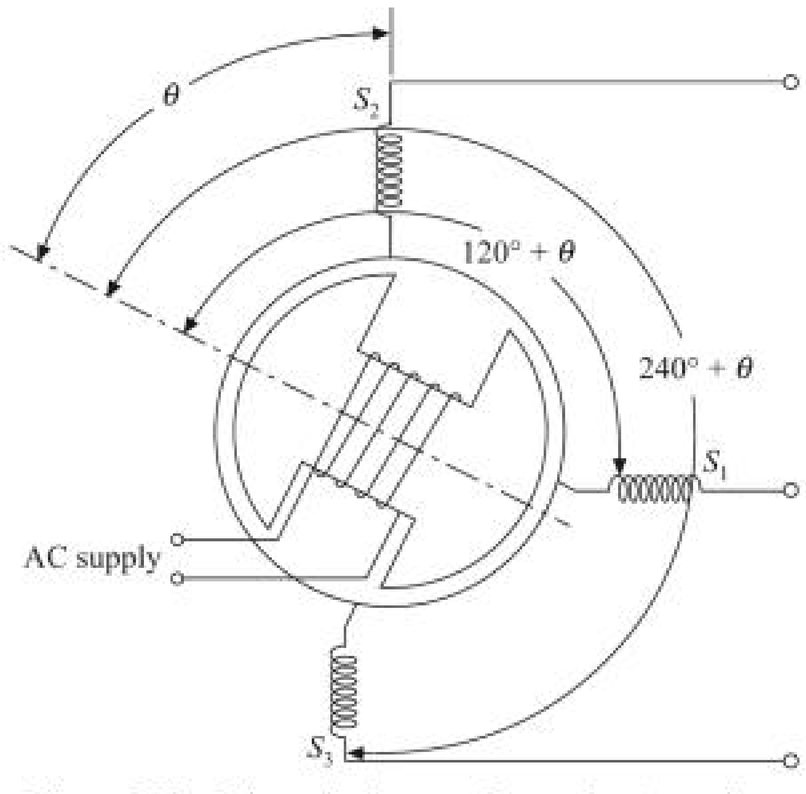


Figure 2.63 Schematic diagram of a synchro transmitter.

Let v_{s1n}, v_{s2n} and v_{s3n} respectively be the voltages in the stator coils S_1, S_2 and S_3 with respect to the neutral. Then, for the rotor position of the synchro transmitter shown in Figure 2.63, where the rotor axis makes an angle θ with the axis of the stator coil S_2,

$$\left.\begin{aligned} v_{s1n} &= KV_r \sin \omega_c t \cos (\theta + 120°) \\ v_{s2n} &= KV_r \sin \omega_c t \cos \theta \\ v_{s3n} &= KV_r \sin \omega_c t \cos (\theta + 240°) \end{aligned}\right\} \tag{2.26}$$

The three terminal voltages of the stator are as follows:

$$\left.\begin{aligned} v_{s1s2} &= v_{s1n} - v_{s2n} = \sqrt{3}\, KV_r \sin (\theta + 240°) \sin \omega_c t \\ v_{s2s3} &= v_{s2n} - v_{s3n} = \sqrt{3}\, KV_r \sin (\theta + 120°) \sin \omega_c t \\ v_{s3s1} &= v_{s3n} - v_{s1n} = \sqrt{3}\, KV_r \sin \theta \sin \omega_c t \end{aligned}\right\} \tag{2.27}$$

when $\theta = 0$, from the above equations [Eqs. (2.26) and (2.27)] it is seen that maximum voltage is induced in the stator coil S_2, while the terminal voltage v_{s3s1} is zero. This position of the rotor is defined as the electrical zero of the transmitter and is used as reference for specifying the angular position of the rotor.

Thus it is seen that the input to the synchro transmitter is the angular position of its rotor shaft and the output is a set of three stator coil to coil single-phase voltages. The magnitudes of these voltages are functions of the shaft position. So by measuring and identifying the set of voltages at the stator terminals, it is possible to identify the angular position of the rotor.

2.8.2 Synchro Control Transformer

The constructional features of a synchro control transformer are similar to that of a synchro transmitter except for the shape of the rotor. The rotor of the control transformer is made cylindrical in shape so that the air gap is practically uniform. This feature of the control transformer minimizes the changes in the rotor impedance with the rotation of the shaft. Another distinguishing feature is that the stator winding of the control transformer has a higher impedance per phase. This feature permits several control transformers to be fed from a single transmitter. The output of the synchro transmitter is applied to the stator windings of a synchro control transformer.

Working: The rotor shaft is connected to the load whose position has to be maintained at the desired value. Depending on the current position of the rotor and the applied emf on the stator, an emf is produced in the rotor winding. This emf can be measured and used to drive a motor so that the position of the load is corrected.

2.8.3 Synchro as an Error Detector

The synchro error detector is formed by interconnection of the synchro transmitter and the synchro control transformer as shown in Figure 2.64. The stator leads of the transmitter are directly connected to the stator leads of the control transformer. The angular position of the transmitter rotor is the reference input and the rotor is excited by a single-phase ac supply. The control transformer rotor is connected to a servomotor and to the shaft of the load, whose position is the desired output. Initially, the transmitter and the control transformer rotor are assumed to be in their electrical zero position. The angular separation of both rotor axes in this position is 90°. The electrical zero position of a control transformer in a servo system is that position of its rotor for which the output voltage on the rotor winding is zero with the transmitter in its electrical zero position.

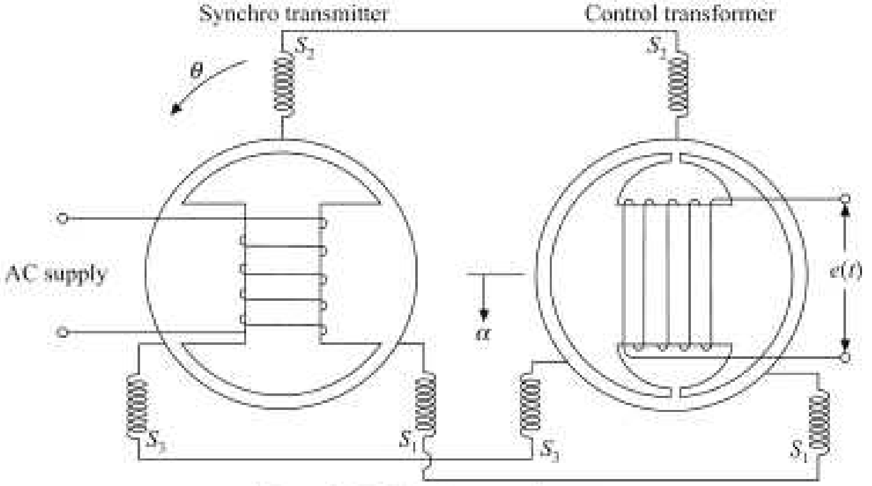


Figure 2.64 Synchro error detector.

When the transmitter rotor is excited, the rotor flux is set up and emfs are induced in stator coils. These induced emfs are impressed on the stator coils of the control transformer. The currents in the stator coils set up flux in the control transformer. Due to the similarity in the magnetic construction, the flux pattern produced in the two synchros will be the same if all losses are neglected.

Let the rotor of the transmitter rotate through an angle θ from its electrical zero position in the direction indicated and let the control transformer rotate in the same direction through an angle α resulting in a net angular separation of $\phi = (90 - \theta + \alpha)$ between the two rotors. So the voltage of the rotor terminals of the control transformer is then

$$e(t) = K'V_r \cos (90 - \theta + \alpha) \sin \omega_c t = K'V_r \cos (\theta - \alpha) \sin \omega_c t \qquad (2.28)$$

Equation (2.28) shows that the output voltage of the synchro error detector is a modulated signal with carrier frequency ω_c.

The synchro transmitter-control transformer pair thus acts as an error detector giving a voltage signal at the rotor terminals of the control transformer proportional to the angular difference between the transmitter and control transformer shaft positions.

2.9 GEAR TRAINS

Gear trains are used in control systems to attain the mechanical matching of the motor to the load. Usually, a servomotor operates at a high speed but at a low torque. To drive a load with a high torque and low speed by such a motor, the torque magnification and speed reduction are achieved by gear trains. Thus in mechanical systems, gear trains act as matching devices like transformers in electrical systems.

Figure 2.65 shows a motor driving a load through a gear train which consists of two gears coupled together. The gear with N_1 teeth is called the *primary gear* (analogous to the primary winding of the transformer) and the gear with N_2 teeth is called the *secondary gear*.

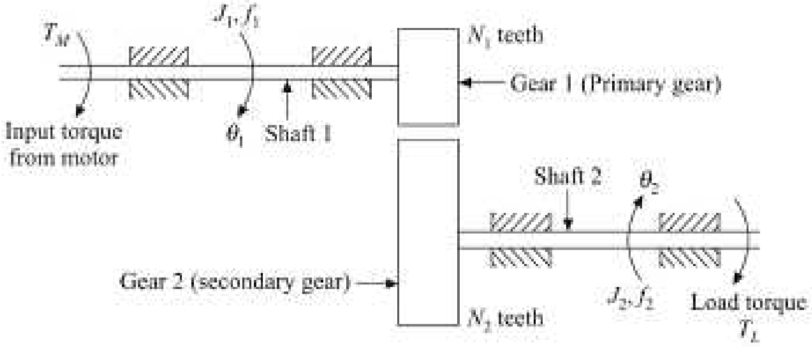


Figure 2.65 Gear train system.

The angular displacements of shafts 1 and 2 are denoted by θ_1 and θ_2 respectively. The moment of inertia and viscous friction of motor and gear 1 are denoted by J_1 and f_1 and those of gear 2 and load by J_2 and f_2 respectively.

For the first shaft, the differential equation is

$$T_M = J_1 \frac{d^2\theta_1}{dt^2} + f_1 \frac{d\theta_1}{dt} + T_1 \tag{2.29}$$

where T_M is the torque developed by the motor and T_1 is the load torque on gear 1 due to the rest of the gear train.

For the second shaft

$$J_2 \frac{d^2\theta_2}{dt^2} + f_2 \frac{d\theta_2}{dt} + T_L = T_2 \tag{2.30}$$

where T_L is the load torque and T_2 is the load torque on gear 2 due to the rest of the gear train.

Let r_1 be the radius of gear 1 and r_2 be the radius of gear 2. Since the linear distance travelled along the surface of each gear is the same, $r_1\theta_1 = r_2\theta_2$. The number of teeth on gear surface being proportional to gear radius, we obtain

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} \tag{2.31}$$

Here the stiffness of the shafts of the gear train is assumed to be infinite. In an ideal case of no loss in power transfer, the work done by gear 1 is equal to that of gear 2. Therefore,

$$T_1\theta_1 = T_2\theta_2$$

$$\therefore \qquad \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} \tag{2.32}$$

Differentiating twice, we have the following relation for speed and acceleration.

$$\frac{\ddot{\theta}_2}{\ddot{\theta}_1} = \frac{\dot{\theta}_2}{\dot{\theta}_1} = \frac{N_1}{N_2} \tag{2.33}$$

Thus, if $\frac{N_1}{N_2} < 1$, the gear train reduces the speed and magnifies the torque.

Eliminating T_1 and T_2 from the torque equation using Eq. (2.32) and then Eq. (2.30), we get

$$T_M = J_1 \frac{d^2\theta_1}{dt^2} + f_1 \frac{d\theta_1}{dt} + T_1 = J_1 \frac{d^2\theta_1}{dt^2} + f_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2} T_2$$

$$J_1 \frac{d^2\theta_1}{dt^2} + f_1 \frac{d\theta_1}{dt} + \frac{N_1}{N_2}\left(J_2 \frac{d^2\theta_2}{dt^2} + f_2 \frac{d\theta_2}{dt} + T_L \right) = T_M \tag{2.34}$$

Eliminating θ_2 using Eq. (2.32), Eq. (2.34) reduces to

$$\left[J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2 \right] \frac{d^2\theta_1}{dt^2} + \left[f_1 + \left(\frac{N_1}{N_2}\right)^2 f_2 \right] \frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right) T_L = T_M$$

Thus, the equivalent moment of inertia and viscous friction of gear train referred to shaft 1 are

$$J_{1eq} = J_1 + \left(\frac{N_1}{N_2}\right)^2 J_2; \quad f_{1eq} = f_1 + \left(\frac{N_1}{N_2}\right)^2 f_2$$

In terms of equivalent moment of inertia and friction, the equation for T_M, i.e. Eq. (2.29) may be written as

$$J_{1eq} \frac{d^2\theta_1}{dt^2} + f_{1eq} \frac{d\theta_1}{dt} + \left(\frac{N_1}{N_2}\right) T_L = T_M$$

where $\left(\frac{N_1}{N_2}\right) T_L$ is the load torque referred to shaft 1. Similarly, expressing θ_1 in terms of θ_2, the equivalent moment of inertia and viscous friction of gear train referred to load shaft are

$$J_{2eq} = J_2 + \left(\frac{N_2}{N_1}\right)^2 J_1; \quad f_{2eq} = f_2 + \left(\frac{N_2}{N_1}\right)^2 f_1$$

The torque equation referred to the load shaft may then be expressed as

$$J_{2eq} \frac{d^2\theta_2}{dt^2} + f_{2eq} \frac{d\theta_2}{dt} + T_L = \left(\frac{N_2}{N_1}\right) T_M$$

SHORT QUESTIONS AND ANSWERS

1. Most control systems contain which elements?

A. Most control systems contain mechanical as well as electrical components.

2. From a mathematical view point, the descriptions of which systems are analogous?

A. From a mathematical view point, the descriptions of mechanical and electrical systems are analogous. Given an electrical device, there is always an analogous mechanical counterpart mathematically and vice versa.

3. What are the two types of mechanical systems?

A. The two types of mechanical systems are (a) mechanical translational systems and (b) mechanical rotational systems.

4. What do you mean by a mechanical translational system?

A. A mechanical translational system is one in which the motion takes place along a straight line.

5. What do you mean by a mechanical rotational system?

A. A mechanical rotational system is one in which the motion takes place about a fixed axis.

6. Using which law, the equations of motion of mechanical systems are formulated?

A. Using Newton's law of motion, the equations governing the motion of mechanical systems are often formulated directly or indirectly.

7. State Newton's law of motion for mechanical translational systems.

A. Newton's law of motion for mechanical translational systems states that the algebraic sum of the forces acting on a rigid body in a given direction is equal to the product of the mass of the body and its acceleration in the same direction.

8. State Newton's law of motion for mechanical rotational systems.

A. Newton's law of motion for mechanical rotational systems states that the algebraic sum of moments or torques about a fixed axis is equal to the product of the inertia and the angular acceleration about the axis.

9. Using which ideal elements the mechanical translational systems are modelled?

A. The mechanical translational systems are modelled by using three ideal elements: mass, spring and damper.

10. Using which ideal elements the mechanical rotational systems are modelled?

A. The mechanical rotational systems are modelled by using three ideal elements: inertia, torsional spring and friction.

11. Define mass.

A. Mass is considered as a property of an element that stores the kinetic energy of translational motion.

12. Which type of energy does a spring store?

A. A spring is considered to be an element that stores potential energy.

13. When does friction exist?

A. Whenever there is motion or tendency of motion between two physical elements, frictional forces exist.

14. What are the three types of friction? Which one is more predominant?

A. The three types of friction commonly used in practical systems are—viscous friction, static friction, and coulomb friction. In most physical systems, the viscous friction predominates.

15. What do you mean by coulomb friction?

A. The force of sliding friction between dry surfaces is called coulomb friction. This force is substantially constant.

16. What do you mean by viscous friction?

A. The force of friction between moving surfaces separated by viscous fluid or the force between a solid body and a fluid medium is called viscous friction. This force is approximately linearly proportional to velocity over a certain limited velocity range.

17. What do you mean by stiction friction?

A. The force required to initiate motion between two contacting surfaces is called stiction.

18. Which friction is represented by a dash pot?

A. Viscous friction is always represented by a dash pot.

19. Define inertia.

A. Inertia J is considered to be the property of an element that stores the kinetic energy of rotational motion.

20. What does torsional spring constant K represent?

A. The torsional spring constant K represents the compliance of a rod or a shaft when it is subjected to an applied torque.

21. What do you mean by a free-body diagram?

A. A free-body diagram is a diagram showing each mass separately, with all forces acting on the mass marked on it.

22. What do you mean by a mechanical network?

A. A mechanical network is a diagram indicating the interconnection of sources and components of the mechanical system in which one end of the force or torque sources and the mass or inertia elements is connected to ground.

23. Write the force balance equations for an (a) ideal mass, (b) ideal spring and (c) ideal dash pot.

A. The force balance equations for the elements of mechanical translational system are as follows:

Mass (M) $\rightarrow F = M\dfrac{d^2x}{dt^2}$ or $F = M\dfrac{dv}{dt}$

Spring (K) $\rightarrow F = Kx$ or $F = K\int v\,dt$

Damper (f) $\rightarrow F = f\dfrac{dx}{dt}$ or $F = fv$

24. Write the torque balance equations for an (a) ideal inertia, (b) ideal spring and (c) ideal dash pot.

A. The torque balance equations for the elements of the mechanical rotational systems are as follows:

Inertia (J) $\rightarrow T = J\dfrac{d^2\theta}{dt^2}$ or $T = J\dfrac{d\omega}{dt}$

Torsional spring (K) $\rightarrow T = K\theta$ or $T = K\int \omega\, dt$

Damper (f) $\rightarrow T = f\dfrac{d\theta}{dt}$ or $T = f\omega$

25. Name the energy storage elements in electrical systems.

A. The capacitor and the inductor are the energy storage elements in electrical systems.

26. Name the energy dissipative elements in electrical systems.

A. The resistor is the energy dissipative element in electrical systems.

27. What is the systematic way of analyzing mechanical systems?

A. The systematic way of analyzing a mechanical system is to draw a free-body diagram or a mechanical network and then write the differential equations describing them.

28. What do you mean by analogous systems?

A. Systems whose differential equations are of identical form are called analogous systems.

29. What are the two types of analogy between mechanical translational and electrical systems?

A. The two types of analogy between mechanical translational and electrical systems are (a) force-voltage analogy and (b) force-current analogy.

30. What are the two types of analogy between mechanical rotational and electrical systems?

A. The two types of analogy between mechanical rotational and electrical systems are (a) torque-voltage analogy and (b) torque-current analogy.

31. Name the analogous electrical elements in force-voltage analogy for the elements of mechanical translational system?

A. In force-voltage analogy, the analogous quantities in mechanical translational and electrical systems are Force (F)-Voltage (e), Mass (M)-Inductance (L), Friction (f)-Resistance (R), Spring constant (K)-Reciprocal of capacitance ($1/C$), Displacement (x)-Charge (q), Velocity (v)-Current (i).

32. Name the analogous electrical elements in force-current analogy for the elements of mechanical translational system?

A. In force-current analogy, the analogous quantities in mechanical translational and electrical systems are Force (F)-Current (i), Mass (M)-Capacitance (C), Friction (f)-Reciprocal of resistance ($1/R$), Spring constant (K)-Reciprocal of inductance ($1/L$), Displacement (x)-Flux linkage (ϕ), Velocity (v)-Voltage (e).

33. Name the analogous electrical elements in torque-voltage analogy for the elements of mechanical rotational system?

A. In torque-voltage analogy, the analogous quantities in mechanical rotational and electrical systems are Torque (T)-Voltage (e), Inertia (J)-Inductance (L), Friction (f)-Resistance (R), Spring constant (K)-Reciprocal of capacitance ($1/C$), Angular displacement (θ)-Charge (q), Angular velocity (ω)-Current (i).

34. Name the analogous electrical elements in torque-current analogy for the elements of mechanical rotational systems?

A. In torque-current analogy, the analogous quantities in mechanical rotational and electrical systems are Torque (T)-Current (i), Inertia (J)-Capacitance (C), Friction (f)-Reciprocal of resistance ($1/R$), Spring constant (K)-Reciprocal of inductance ($1/L$), Angular displacement (θ)-Flux linkage (ϕ), Angular velocity (ω)-Voltage (e).

35. What do you mean by analogy? What is the advantage of finding analogy between systems?

A. Analogy means similarity. Using analogy, it is convenient to study a non-electrical system in terms of its electrical analog as electrical systems are more easily amenable to experimental study. The advantage of finding analogy between two systems is an unfamiliar system can be analyzed easily in terms of the familiar system.

36. Define the term 'Impulse response of a system'.

A. The impulse response of a system is defined as the output when the input is a unit impulse function $\delta(t)$. Once the impulse response of a linear system is known, the output of the system $c(t)$, with any input $r(t)$ can be found by using the transfer function.

37. Give the two definitions of the transfer function.

A. The transfer function of a linear time-invariant system is defined as the Laplace transform of the impulse response of the system, with all the initial conditions set to zero. The transfer function of a linear time-invariant system is also defined as the ratio of the Laplace transform of the output to the Laplace transform of the input with all initial conditions neglected.

38. Can a transfer function be defined for nonlinear systems?

A. No. A transfer function can not be defined for nonlinear systems.

39. For which systems can a transfer function be defined?

A. The transfer function can be defined only for linear time-invariant systems.

40. What do you mean by a strictly proper transfer function?

A. A strictly proper transfer function is one in which the order of the denominator polynomial is greater than that of the numerator polynomial.

41. What do you mean by a proper transfer function?

A. A proper transfer function is one in which the order of the numerator polynomial is equal to that of the denominator polynomial.

42. What do you mean by an improper transfer function?

A. An improper transfer function is one in which the order of the denominator polynomial is lees than that of the numerator polynomial.

43. What do you mean by characteristic equation? Why that name?

A. The characteristic equation of a linear system is defined as the equation obtained by setting the denominator polynomial of the transfer function to zero. This equation characterizes the behaviour of the system. Hence the name characteristic equation.

44. The stability of a linear time-invariant system is governed by what?

A. The stability of a linear time-invariant system is governed completely by the roots of its characteristic equation.

45. What do you mean by a sinusoidal transfer function?

A. The transfer function obtained by replacing s with $j\omega$ in the original transfer function is called the sinusoidal transfer function.

46. What do you mean by multivariable system?

A. A multi-input-multi-output system is called a multivariable system.

47. Does the transfer function of a system depend on the input?

A. No. The transfer function of a system does not depend on the input.

48. How is the transfer function to be derived when a number of systems are cascaded?

A. When loading is considered, the overall transfer function should be derived by basic analysis without regard to individual transfer functions.

When there is no loading, the overall transfer function is equal to the product of the individual transfer functions of the systems connected in cascade.

49. Why are differential equations converted into algebraic equations?

A. The solution of higher-order differential equations is quite a tedious process, so differential equations are converted into algebraic equations by using Laplace transforms.

50. What do you mean by loading in electrical systems?

A. Loading in electrical systems means drawing the current from them.

51. What are servomotors?

A. The motors used in control systems or in servomechanisms are called servomotors. They are used to convert electrical signal into angular motion.

52. What are the characteristics of servomotors?

A. The characteristics of servomotors are as follows:

1. Linear relationship between the speed and the electric control signal
2. Steady-state stability
3. Wide range of speed control
4. Low mechanical and electrical inertia
5. Fast response

53. Compare ac and dc servomotors.

A. The ac and dc servomotors are compared as follows:

ac servomotor	*dc servomotor*
1. Low power output.	1. Relatively high power output than an ac servomotor of the same size.
2. Characteristics are nonlinear.	2. Characteristics are linear.
3. Slow response due to higher values of time constants.	3. Fast response due to low electrical and mechanical time constants.
4. Suitable for low power applications.	4. Suitable for high power applications.

54. What are the advantages of permanent magnet dc servomotors?

A. The advantages of permanent magnet dc servomotors are as follows:

1. A simpler and more reliable motor because the field power supply is not required.
2. Higher efficiency due to the absence of field losses.
3. Field flux is less affected by temperature rise.
4. Less heating, making it possible to totally enclose the motor.
5. No possibility of over speeding due to loss of field.
6. A more linear torque versus speed curve.
7. Higher power output at the same dimensions and temperature limitations.

55. What are the special features of electromagnetic field dc servomotors?

A. The special features of electromagnetic field dc servomotors are as follows:

1. The number of slots and commutator segments is large to improve commutation.
2. Compoles and compensating windings are provided to eliminate sparking.
3. The diameter to length ratio is kept low to reduce inertia.
4. Oversize shafts are employed to withstand the high torque stress.
5. Eddy currents are reduced by complete lamination of the magnetic circuit and by using low-loss steel.

56. What for dc motors are used in control systems?

A. In servo applications, dc motors are used to produce rapid acceleration from stand still. So the physical requirements of such motors are low inertia and high starting torque.

57. What are the two control modes in which the dc motors are used in control systems?

A. The two control modes in which the dc motors are used in control systems are (a) armature control mode with fixed field current and (b) field control mode with fixed armature current.

58. For which motors field control is preferred and for which motors armature control is preferred?

A. For small size motors, field control is advantageous and for large size motors armature control is preferred.

59. How does an ac servomotor differ from a normal two-phase induction motor?

A. An ac servomotor is basically a two-phase induction motor but it differs from that in two ways.

1. The ac servomotor has low value of X/R to achieve linear torque-speed characteristics. But normal induction motors have large values of X/R for higher efficiency.
2. The ac servomotor has low inertia rotor. The inertia of the rotor is reduced by reducing the diameter or by drag-cup construction.

60. What is a synchro?

A. A synchro is an electromagnetic transducer commonly used to convert an angular position of a shaft into an electric signal. It works on the principle of a rotating transformer (induction motor).

61. What are the commercial names of a synchro?

A. The commercial names of a synchro are: selsyn, autosyn and telesyn.

62. What is synchro pair? What for is it used?

A. A synchro pair is a system formed by interconnection of the devices—synchro transmitter and synchro control transformer. A synchro pair may be used either to transmit an angular motion from one place to another or to produce an error voltage proportional to the difference between two angular motions.

63. What are the differences between a synchro transmitter and a synchro control transformer?

A. The differences between a synchro transmitter and a synchro control transformer are as follows:

1. The rotor of a synchro transmitter is of dumb-bell shape, but the rotor of a synchro control transformer is cylindrical.
2. The rotor winding of a synchro transmitter is excited by an ac voltage. In a synchro control transformer, the induced emf in the rotor is used as an output signal (error signal).

64. What is electrical zero position of a synchro transmitter?

A. The electrical zero position of a synchro transmitter is a position of its rotor at which one of the coil-to-coil voltage is zero. Any angular motion of the rotor is measured with respect to the electrical zero position of the rotor.

65. What is electrical zero position of a synchro control transformer?

A. The electrical zero position of a synchro control transformer in a servo system is that position of its rotor for which the output voltage on the rotor winding is zero, with the transmitter in its electrical zero position.

66. What is aligned position of a synchro pair?

A. The aligned position of a synchro pair is one in which both the synchro transmitter and the synchro control transformer are in their electrical zero positions. The angular separation of both rotor axes in aligned position is 90°. The error signal is zero in the aligned position.

67. What are the applications of synchros?

A. Synchros are used in servomechanisms (position control systems) as error detectors and to convert angular displacements to proportional electrical signals. They are also used in control systems to transmit angular motions from one place to another.

68. What for gear trains are used in control systems?

A. Gear trains are used in control systems to obtain mechanical matching of motor to load. They are used to alter the speed to torque ratio of the rotational power transmitted from motor to load. This is necessary to meet the torque requirement of the load.

69. What is gear ratio?

A. The gear ratio of a gear train is the ratio of torque, speed, angular displacement, velocity and acceleration between any two gear wheels in that gear train. The gear ratio between gear wheels a and b in a gear train is given by

$$\frac{N_a}{N_b} = \frac{r_a}{r_b} = \frac{T_a}{T_b} = \frac{\theta_b}{\theta_a} = \frac{\omega_b}{\omega_a} = \frac{\alpha_b}{\alpha_a}$$

70. A servomotor is connected through a gear ratio of 12 to a load having moment of inertia J and viscous friction coefficient f. What are the equivalent parameters referred to motor shaft side.

A. Let N_1 = number of teeth in gear wheel connected to motor shaft

N_2 = number of teeth in gear wheel connected to load shaft

Given $N_1/N_2 = 12$

Moment of inertia of load referred to motor shaft = $J(N_1/N_2)^2 = 144\ J$

Coefficient of friction of load referred to motor shaft = $f(N_1/N_2)^2 = 144\ f$

71. How do you compare gear trains and transformers?

A. In mechanical systems, gear trains act as matching devices like transformers in electrical systems.

REVIEW QUESTIONS

1. Derive an expression for the transfer function of a single-loop control system.
2. Derive an expression for the transfer function of an armature controlled dc servomotor.
3. Derive an expression for the transfer function of a field controlled dc servomotor.
4. Explain the construction and working of an ac servomotor.
5. Derive an expression for the transfer function of an ac servomotor.
6. Explain the construction and working of a synchro.
7. Write short notes on gear trains. Derive an expression for the motor torque.

FILL IN THE BLANKS

1. Most control systems contain _________ as well as ________ components.
2. From a mathematical view point, the descriptions of mechanical and electrical elements are _______________.
3. The motion of mechanical elements can be described as ________ or ________ or _______.
4. In mechanical ________ systems, motion takes place along a straight line.
5. In mechanical _________ systems, motion takes place about a fixed axis.
6. The variables that are used to describe the translational motion are _______ ______ ______.
7. ____________ is considered as a property of an element that stores kinetic energy of translational motion.
8. In general _______ is an element that stores potential energy.
9. The force of sliding friction between dry surfaces is called _______ friction force.
10. The force required to initiate motion between two contacting surfaces is called ______.
11. The element for viscous friction is often represented by a __________.
12. The force of friction between moving surfaces separated by a fluid or the force between a solid body and a fluid medium is called __________ force.
13. ___________ is considered as a property of an element that stores the kinetic energy of rotational motion.
14. Systems whose differential equations are identical are called ______ systems.
15. The classical way of modelling linear systems is to use _________ to represent input-output relations between variables.
16. The transfer function is defined only for _________ systems. It is not defined for ______ systems.
17. The transfer function is defined only when the ________ are neglected.
18. The transfer function is independent of the _________ to the system.
19. The transfer function between an input variable and an output variable of a system is defined as the Laplace transform of the _______________ of the system.
20. The transfer function is defined as the ratio of the _________ to the _________ with the assumption that all _________ are zero.
21. The transfer function is said to be __________ if the order of the denominator polynomial is greater than that of the numerator polynomial.
22. The transfer function is said to be _________ if the order of the numerator polynomial is equal to that of the denominator polynomial.

23. The transfer function is said to be __________ if the order of the numerator polynomial is greater than that of the denominator polynomial.
24. The characteristic equation of a linear system is defined as the equation obtained by setting the __________ of the ___________ to zero.
25. The transfer function obtained by replacing s with $j\omega$ in the original transfer function is called the ___________.
26. In servo applications ___________ are required to produce rapid acceleration from stand still.
27. In control systems, the dc motors are used in two different control modes (a) ________ and (b) __________.
28. In servo applications, the dc motors are generally used in the _________ range of the magnetization curve.
29. For small size motors ________ is advantageous and for large size motors _________ is preferred.
30. Gear trains are used in control systems to attain mechanical matching of _________ to _________.
31. In mechanical systems, gear trains act as matching devices like ________ in electrical systems.
32. The motors used in automatic control systems or in servomechanisms are called __________.
33. Servomotors are used to convert _________ into ___________.
34. The characteristics of _________ servomotors are nonlinear, whereas the characteristics of __________ servomotors are linear.
35. __________ servomotor is suitable for low power applications and __________ servomotor is suitable for high power applications.
36. _________ servomotor has fast response and _________ servomotor has slow response.
37. The ac servomotor has __________ value of X/R, whereas a normal induction motor has a __________ value of X/R.
38. A _________ is an electro magnetic transducer commonly used to convert an angular position of a shaft into an electric signal.
39. A synchro is commonly known as __________ or __________.
40. A synchro pair consists of a ____________ and a ___________.
41. The rotor of synchro transmitter is of __________ shape, but the shape of synchro control transformer is __________.
42. Any angular motion of the rotor of a transmitter is measured with respect to the _________ of the rotor.

Answers to Fill in the Blanks

1. mechanical, electrical **2.** analogous **3.** translational, rotational, combinations of both **4.** translational **5.** rotational **6.** displacement, velocity, acceleration **7.** mass **8.** spring **9.** coulomb **10.** stiction **11.** dash pot **12.** viscous friction **13.** inertia **14.** analogous **15.** transfer function **16.** linear time-invariant, nonlinear **17.** initial conditions **18.** input **19.** impulse response **20.** Laplace transform of the output, Laplace transform of the input, initial conditions **21.** strictly proper **22.** proper **23.** improper **24.** denominator polynomial, transfer function **25.** sinusoidal transfer function **26.** dc motors **27.** (a) armature control mode with fixed field current, (b) field control mode with fixed armature current **28.** linear **29.** field control, armature control **30.** motor, load **31.** transformers **32.** servomotors **33.** electrical signal, angular motion **34.** ac, dc **35.** ac, dc **36.** dc, ac **37.** low, large **38.** synchro **39.** selsyn, autosyn **40.** Synchro transmitter, synchro control transformer **41.** Dumb-bell, cylindrical **42.** Electrical zero position.

OBJECTIVE TYPE QUESTIONS

1. The motion of mechanical elements can be described as
   (a) purely rotational
   (b) purely translational
   (c) rotational or translational or combination of both
   (d) none of these

2. Translational motion is the motion
   (a) along a straight line   (b) about a fixed axis
   (c) along a random path   (d) none of these

3. Rotational motion is the motion
   (a) along a straight line   (b) about a fixed axis
   (c) along a random path   (d) none of these

4. An element that stores the kinetic energy of translational motion is called
   (a) mass   (b) spring
   (c) damper   (d) none of these

5. An element which stores potential energy is
   (a) mass   (b) spring
   (c) damper   (d) none of these

6. The force of sliding friction between dry surfaces is called
   (a) coulomb friction force   (b) viscous friction force
   (c) stiction   (d) none of these

7. The force required to initiate motion between two contacting surfaces is called
   (a) coulomb friction force   (b) viscous friction force
   (c) stiction   (d) none of these

8. The friction force acts in a direction
(a) opposite to that of motion
(b) perpendicular to that of motion
(c) along that of motion
(d) none of these

9. The viscous friction force is approximately linearly proportional to
(a) displacement
(b) velocity
(c) acceleration
(d) none of these

10. An element that stores kinetic energy of rotational motion is called
(a) inertia
(b) torsional spring
(c) damper
(d) mass

11. The transfer function is defined only for
(a) linear time-varying systems
(b) linear time-invariant systems
(c) linear and nonlinear systems
(d) all of these

12. The transfer function is defined as the Laplace transform of the response for a
(a) step input
(b) impulse input
(c) ramp input
(d) parabolic input

13. The transfer function is dependent of the
(a) parameters of the system
(b) initial conditions of the system
(c) input to the system
(d) output of the system

14. For strictly proper transfer function, the order of the numerator is
(a) equal to that of the denominator
(b) greater than that of the denominator
(c) smaller than that of the denominator
(d) not related to that of the denominator

15. For proper transfer function, the order of the numerator is
(a) equal to that of the denominator
(b) greater than that of the denominator
(c) smaller than that of the denominator
(d) not related to that of the denominator

16. For an improper transfer function, the order of the numerator is
(a) equal to that of the denominator
(b) greater than that of the denominator
(c) smaller than that of the denominator
(d) not related to that of the denominator

Answers to Objective Type Questions

1. c **2.** a **3.** b **4.** a **5.** b **6.** a **7.** c **8.** a **9.** b **10.** a **11.** b **12.** b **13.** a **14.** c **15.** a **16.** b

PROBLEMS

2.1 *to* 2.12: (a) Write the differential equations governing the behaviour of the mechanical systems shown below.
(b) Draw the analogous electrical circuits based on force-current and force-voltage analogies.
(c) Write the corresponding differential equations.

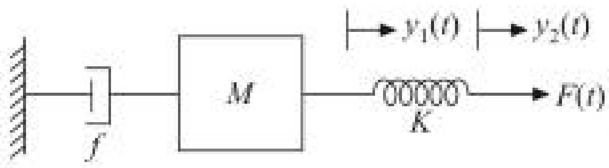


Figure P2.1

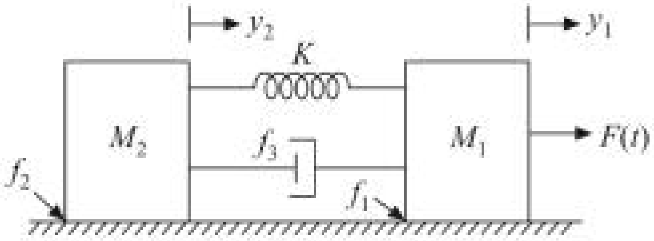


Figure P2.2

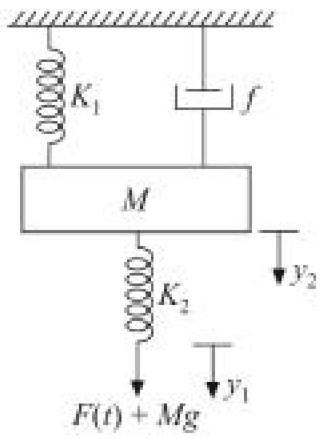


Figure P2.3

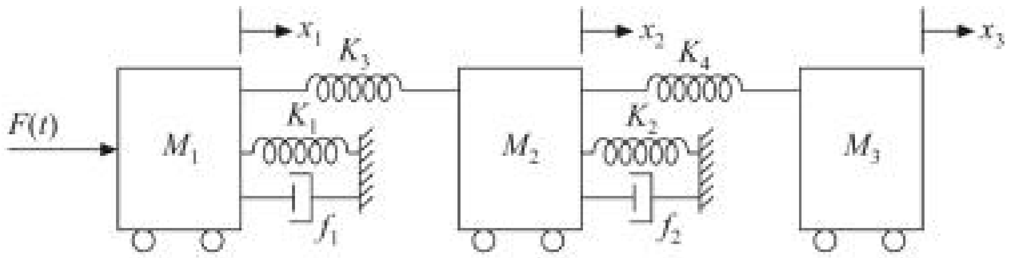


Figure P2.4

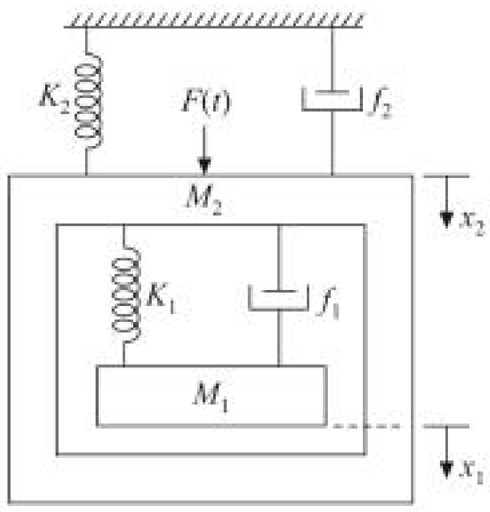


Figure P2.5

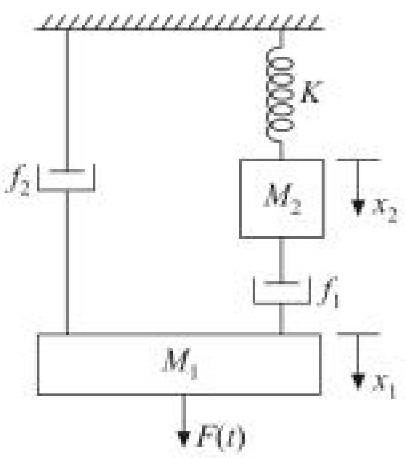


Figure P2.6

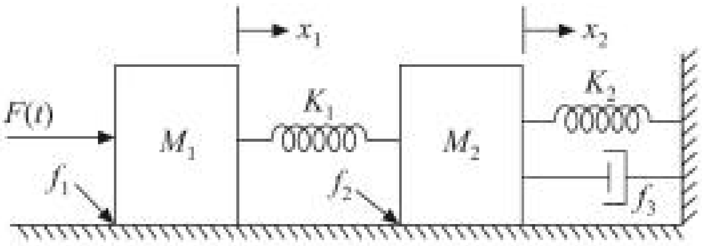


Figure P2.7

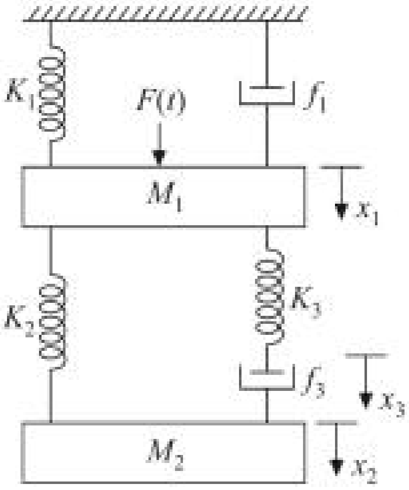


Figure P2.8

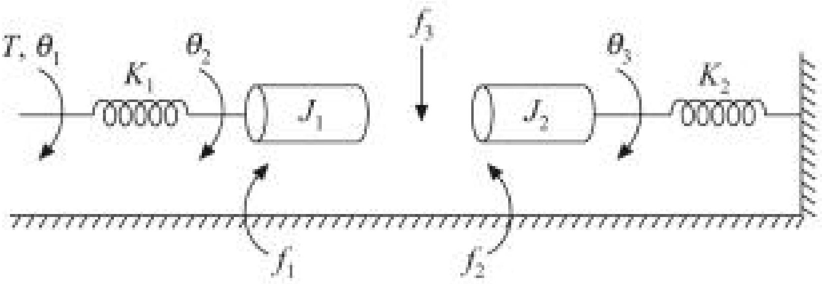


Figure P2.9

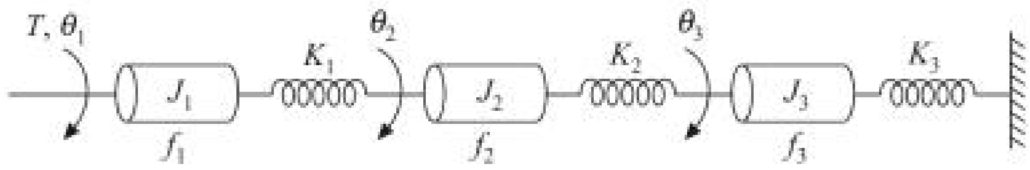


Figure P2.10

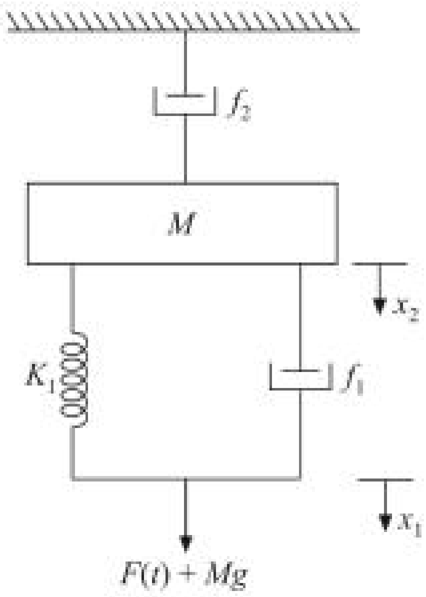


Figure P2.11

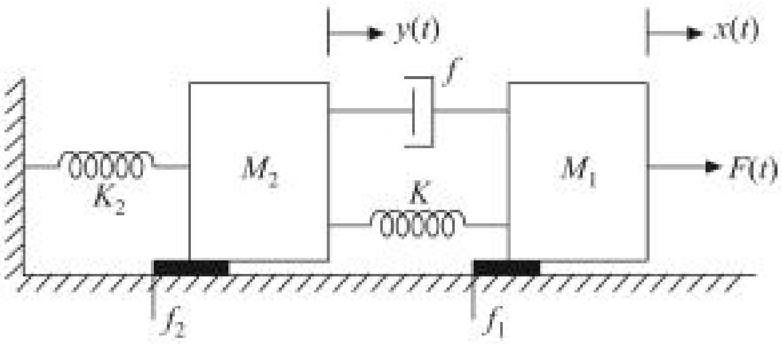


Figure P2.12

Answers to Problems

2.1. (a) $F(t) = K(y_2 - y_1),\ M\ddot{y}_1 + f\dot{y}_1 + K(y_1 - y_2) = 0$

2.2. (a) $F(t) = M_1\ddot{y}_1 + f_1\dot{y}_1 + K(y_1 - y_2) + f_3(\dot{y}_1 - \dot{y}_2) = 0$

$M_2\ddot{y}_2 + f_3(\dot{y}_2 - \dot{y}_1) + K(y_2 - y_1) + f_2\dot{y}_2 = 0$

2.3. (a) $F(t) + Mg = K_2(y_1 - y_2),\ M\ddot{y}_2 + K_2(y_2 - y_1) + f\dot{y}_2 + K_1y_2 = 0$

2.4. (a) $F(t) = M_1\ddot{x}_1 + K_3(x_1 - x_2) + K_1x_1 + f_1\dot{x}_1,$

$M_2\ddot{x}_2 + K_3(x_2 - x_1) + K_4(x_2 - x_3) + K_2x_2 + f_2\dot{x}_2 = 0$

$M_3\ddot{x}_3 + K_4(x_3 - x_2) = 0$

2.5. (a) $F(t) = M_2\ddot{x}_2 + f_2\dot{x}_2 + K_2x_2 + K_1(x_2 - x_1) + f_1(\dot{x}_2 - \dot{x}_1) = 0,$

$M_1\ddot{x}_1 + f_1(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2) = 0$

2.6. (a) $F(t) = M_1\ddot{x}_1 + f_1(\dot{x}_1 - \dot{x}_2) + f_2\dot{x}_1 = 0,\quad M_2\ddot{x}_2 + f_1(\dot{x}_2 - \dot{x}_1) + Kx_2 = 0$

2.7. (a) $F(t) = M_1\ddot{x}_1 + f_1\dot{x}_1 + K_1(x_1 - x_2),\quad M_2\ddot{x}_2 + f_2\dot{x}_2 + f_3\dot{x}_2 + K_2x_2 + K_1(x_2 - x_1) = 0$

2.8. (a) $F(t) = M_1\ddot{x}_1 + f_1\dot{x}_1 + K_1x_1 + K_3(x_1 - x_3) + K_2(x_1 - x_2),$

$M_2\ddot{x}_2 + f_3(\dot{x}_2 - \dot{x}_3) + K_2(x_2 - x_1) = 0,\ K_3(x_3 - x_1) + f_3(\dot{x}_3 - \dot{x}_2) = 0$

2.9. (a) $T = K_1(\theta_1 - \theta_2),\ J_1\ddot{\theta}_2 + K_1(\theta_2 - \theta_1) + f_1\dot{\theta}_2 + f_3(\dot{\theta}_2 - \dot{\theta}_3) = 0,$

$J_2\ddot{\theta}_3 + f_2\dot{\theta}_3 + f_3(\dot{\theta}_3 - \dot{\theta}_2) + K_2\theta_3 = 0$

2.10. (a) $T = J_1\ddot{\theta}_1 + f_1\dot{\theta}_1 + K_1(\theta_1 - \theta_2),\ J_2\ddot{\theta}_2 + f_2\dot{\theta}_2 + K_1(\theta_2 - \theta_1) + K_2(\theta_2 - \theta_3) = 0,$

$J_3\ddot{\theta}_3 + f_3\dot{\theta}_3 + K_2(\theta_3 - \theta_2) + K_3\theta_3 = 0$

2.11. (a) $F(t) + Mg = f_1(\dot{x}_1 - \dot{x}_2) + K_1(x_1 - x_2),$

$M\ddot{x}_2 + f_1(\dot{x}_2 - \dot{x}_1) + K_1(x_2 - x_1) + f_2\dot{x}_2 = 0$

2.12. (a) $F(t) = M_1\ddot{x} + f_1\dot{x} + f(\dot{x} - \dot{y}) + K(x - y),$

$M_2\ddot{y} + f_2\dot{y} + f(\dot{y} - \dot{x}) + K(y - x) + K_2y = 0$

3

Block Diagram and Signal Flow Graphs

3.1 BLOCK DIAGRAMS

A control system may consist of a number of components. To show the functions performed by each component in control engineering, we commonly use a diagram called the block diagram. *A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.* Such a diagram depicts the inter relationships that exist among the various components. It can be used together with the transfer functions to describe the cause-and-effect relationships throughout the system. Differing from a purely abstract mathematical representation, a block diagram has the advantage of indicating more realistically the signal flows of the actual system. In general, block diagrams can be used to model linear as well as nonlinear systems.

In a block diagram, all system variables are linked to each other through functional blocks. The functional block or simply block is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. The signal can pass only in the direction of the arrows. Thus, the block diagram of a system explicitly shows a unilateral property.

Figure 3.1 shows an element of the block diagram. The arrow head pointing towards the block indicates the input, and the arrow head leading away from the block represents the output. Such arrows are referred to as signals.

Figure 3.1 Element of a block diagram.

The dimensions of the output signals from the block are the dimensions of the input signal to the block multiplied by the dimensions of the transfer function in the block.

The advantages of the block diagram representation of a system lie in the fact that it is easy to form the overall block diagram of the entire system by merely connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

In general, the functional operation of the system can be visualized more readily by examining the block diagram than by examining the physical system itself. A block diagram contains information regarding dynamic behaviour, but it does not include any information on the physical construction of the system. Consequently many dissimilar and unrelated systems can be represented by the same block diagram.

In a block diagram, the main source of energy is not explicitly shown and the block diagram of a system is not unique. A number of different block diagrams can be drawn for a system, depending on the point of view of analysis.

The plus or minus sign at each arrow head indicates whether the signal is to be added or subtracted. It is important that the quantities being added or subtracted have the same dimensions and the same units.

The block diagram elements used frequently in control systems and the related algebra are shown in Figure 3.2. One of the important components of a control system is the sensing device that acts as a junction point for signal comparisons. The physical components involved are: the potentiometer, synchro, resolver, differential amplifier, multiplier and other signal processing transducers. In general, sensing devices perform simple mathematical operations such as addition, subtraction, multiplication (nonlinear) and some times combination of these.

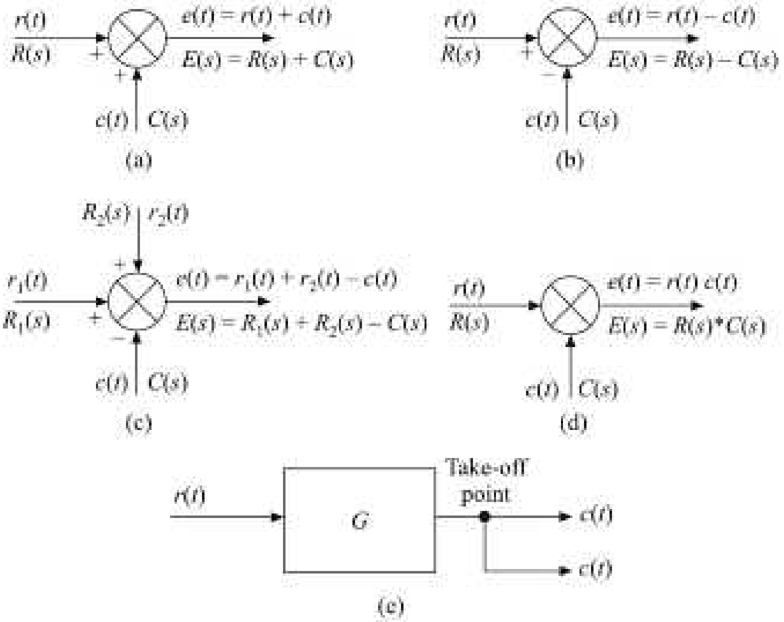


Figure 3.2 (a), (b) and (c) Addition and subtraction operations and (d) multiplication operation and (e) take-off point.

The block diagram representations of these operations are illustrated in Figure 3.2. In Figure 3.2, a circle with a cross is the symbol that indicates a summing operation. The addition and subtraction operations in Figure 3.2(a), Figure (b) and Figure (c) are linear, so that the input and output variables of these block diagram elements can be time-domain variables or Laplace transform variables.

$$e(t) = r(t) - y(t)$$
$$E(s) = R(s) - Y(s)$$

The multiplication operation is nonlinear, so that the input-output relation has meaning only in the real (time) domain.

Branch point: A branch point or a take-off point is a point from which the signal from a block goes concurrently to other blocks or summing points.

3.1.1 Block Diagram of a Closed-Loop System

Figure 3.3 shows an example of a block diagram of a closed-loop system. The output $C(s)$ is fed back to the summing point where it is compared with the reference input. When the output is fed back to the summing point for comparison with the input, it is necessary to convert the form of the output signal to that of the input signal. The role of the feedback element is to modify the output before it is compared with the input. This conversion is accomplished by the feedback element whose transfer function is $H(s)$. The output of the block $C(s)$ in this case is obtained by multiplying the transfer function $G(s)$ by the input to the block $E(s)$. The feedback signal that is fed back to the summing point for comparison with the input is $B(s) = C(s)H(s)$.

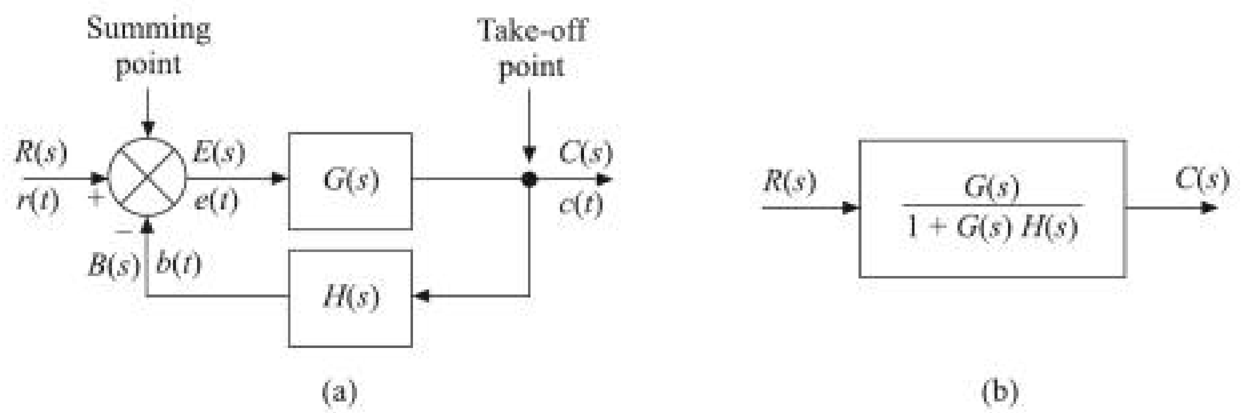


Figure 3.3 Block diagram of a closed-loop control system.

The following terminology is defined with reference to the diagram of Figure 3.3:

$r(t)$, $R(s)$ = reference input (command)
$c(t)$, $C(s)$ = output (controlled variable)
$b(t)$, $B(s)$ = feedback signal
$e(t)$, $E(s)$ = error signal
$G(s)$ = forward path transfer function
$H(s)$ = feedback path transfer function

$G(s)\,H(s) = L(s)$ = loop transfer function or open-loop transfer function

$$T(s) = \frac{C(s)}{R(s)} = \text{closed-loop transfer function or system transfer function}$$

The closed-loop transfer function $T(s)$ can be expressed as a function of $G(s)$ and $H(s)$. From Figure 3.3,

$$C(s) = G(s)\,E(s) \tag{3.1}$$

and

$$B(s) = C(s)\,H(s) \tag{3.2}$$

The error signal is

$$E(s) = R(s) - B(s) \tag{3.3}$$

Substituting Eqs. (3.3) and (3.2) in Eq. (3.1), we get

$$\begin{aligned} C(s) &= G(s)[R(s) - B(s)] \\ &= G(s)R(s) - G(s)B(s) \\ &= G(s)R(s) - G(s)H(s)C(s) \end{aligned}$$

or

$$C(s)[1 + G(s)H(s)] = G(s)R(s) \tag{3.4}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

In general, a control system may contain more than one feedback loop, and evaluation of transfer function from the block diagram by means of the algebraic method described above may be tedious. The transfer function of any linear system can be obtained directly from its block diagram by use of the signal flow graph gain formula.

3.1.2 Block Diagrams and Transfer Functions of Multivariable Systems

When multiple inputs are present in a linear system, each input can be treated independently of the others. The complete output of the system can then be obtained by superposition, i.e. outputs corresponding to each input alone are added together.

Consider a two-input linear system shown in Figure 3.4(a). The response to the reference input can be obtained by assuming $U(s) = 0$. The corresponding block diagram shown in Figure 3.4(b) gives

$$\begin{aligned} C_R(s) &= \text{output due to } R(s) \text{ acting alone} \\ &= \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} R(s) \end{aligned} \tag{3.5}$$

Similarly, the response to the input $U(s)$ is obtained by assuming $R(s) = 0$. The block diagram corresponding to this case is shown in Figure 3.4 (c) which gives

$$\begin{aligned} C_U(s) &= \text{output due to } U(s) \text{ acting alone} \\ &= \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} U(s) \end{aligned} \tag{3.6}$$

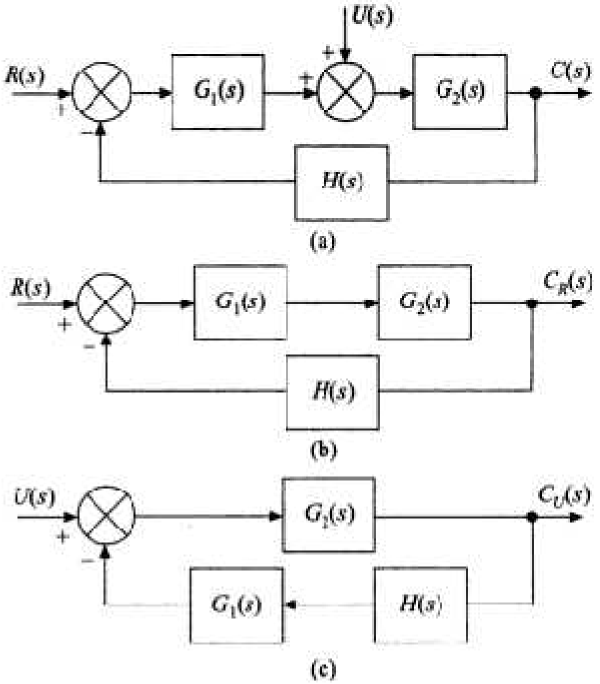


Figure 3.4 (a) Block diagram of a 2-input system, (b) block diagram when $U(s)$ is zero, and (c) block diagram when $R(s)$ is zero.

The response due to the simultaneous application of $R(s)$ and $U(s)$ can be obtained by adding the two individual responses, i.e. Eqs. (3.5) and (3.6).

$$C(s) = C_R(s) + C_U(s)$$

$$= \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}[G_1(s)R(s) + U(s)] \tag{3.7}$$

Consider now the case, where $|G_1(s)H(s)| >> 1$ and $|G_1(s)G_2(s)H(s)| >> 1$. In this case, the closed-loop transfer function $C_U(s)/U(s)$ becomes almost zero, and the effect of the disturbance is suppressed. This is an advantage of the closed-loop system.

On the other hand, the closed-loop transfer function $C_R(s)/R(s)$ approaches $1/H(s)$ as the gain $G_1(s)G_2(s)H(s)$ increases. This means that if $|G_1(s)G_2(s)H(s)| >> 1$, then the closed-loop transfer function $C_R(s)/R(s)$ becomes independent of $G_1(s)$ and $G_2(s)$ and becomes inversely proportional to $H(s)$ so that variations of $G_1(s)$ and $G_2(s)$ do not affect the closed-loop transfer function $C_R(s)/R(s)$. This is another advantage of the closed-loop system. It can easily be seen that any closed-loop system with unity feedback $H(s) = 1$, tends to equalize the input and output.

Two block diagram representations of a multivariable system with p inputs and q outputs are shown in Figure 3.5. In Figure 3.5(a), the individual input and output signals are designated whereas in the block diagram of Figure 3.5(b), the multiplicity of inputs and outputs is denoted by vectors. In practice, the case of Figure 3.5(b) is preferable because of simplicity.

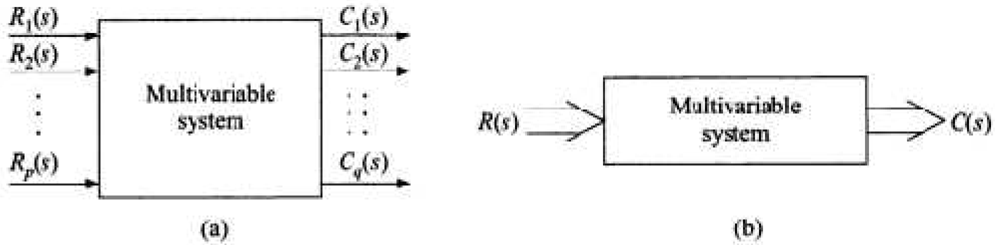


Figure 3.5 Block diagram representation of a multivariable system.

In the case of multiple-input-multiple-output system shown in Figure 3.5(a), the ith output $C_i(s)$ is given by the principle of superposition as

$$C_i(s) = \sum_{j=1}^{r} G_{ij}(s)R_j(s);\ i = 1, 2, \cdots, m \tag{3.8}$$

where $R_j(s)$ is the jth input and $G_{ij}(s)$ is the transfer function between the ith output and jth input with all other inputs reduced to zero.

In matrix form, Eq. (3.8) can be expressed as

$$\begin{bmatrix} C_1(s) \\ C_2(s) \\ \vdots \\ C_q(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) & \cdots & G_{1p}(s) \\ G_{21}(s) & G_{22}(s) & \cdots & C_{2p}(s) \\ \vdots & & & \\ G_{q1}(s) & G_{q2}(s) & \cdots & G_{qp}(s) \end{bmatrix} \begin{bmatrix} R_1(s) \\ R_2(s) \\ \vdots \\ R_p(s) \end{bmatrix}$$

This can be expressed in compressed matrix notation as

$$\boldsymbol{C}(s) = \boldsymbol{G}(s)\boldsymbol{R}(s) \tag{3.9}$$

where, $\boldsymbol{R}(s)$ = vector of inputs (In Laplace transform) (p)

$\boldsymbol{G}(s)$ = Matrix transfer function ($q \times p$)

$\boldsymbol{C}(s)$ = vector of outputs (In Laplace transform) (q)

The corresponding block diagram can be drawn as in Figure 3.5(b), where thick arrows represent multi-inputs and outputs.

Figure 3.6 shows the block diagram of a multivariable feedback control system. When a feedback loop is present as in Figure 3.6, each feedback signal is obtained by processing in general all the outputs. Thus for ith feedback signal, we can write

$$B_i(s) - \sum_{j=1}^{m} H_{ij}(s)C_j(s) \tag{3.10}$$

The ith error signal is then

$$E_i(s) = R_i(s) - B_i(s) \tag{3.11}$$

Generalizing, in matrix form, we can write

$$E_i(s) = R_i(s) - B_i(s) \tag{3.12}$$

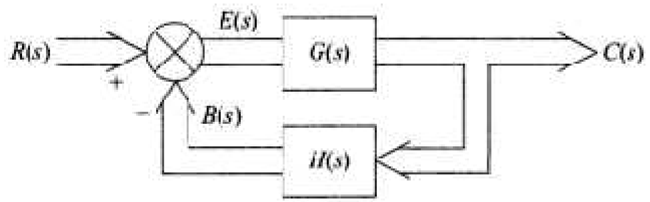


Figure 3.6 Block diagram of a multivariable feedback control system.

The transfer function relationships of the system are expressed in vector-matrix form as

$$\mathbf{C}(s) = \mathbf{G}(s)\mathbf{E}(s)$$

$$\mathbf{E}(s) = \mathbf{R}(s) - \mathbf{B}(s)$$

$$\mathbf{B}(s) = \mathbf{C}(s)\ \mathbf{H}(s)$$

where $\mathbf{C}(s)$ is the $q \times 1$ output vector; $\mathbf{E}(s)$, $\mathbf{R}(s)$, and $\mathbf{B}(s)$ are all $p \times 1$ vectors; and $\mathbf{G}(s)$ and $\mathbf{H}(s)$ are $q \times p$ and $p \times q$ transfer function matrices respectively. Simplifying

$$\mathbf{C}(s) = \mathbf{G}(s)\mathbf{E}(s) = \mathbf{G}(s)[\mathbf{R}(s) - \mathbf{B}(s)]$$

$$= \mathbf{G}(s)\mathbf{R}(s) - \mathbf{G}(s)\mathbf{B}(s)$$

$$= \mathbf{G}(s)\mathbf{R}(s) - \mathbf{G}(s)\mathbf{H}(s)\mathbf{C}(s)$$

$$\therefore \quad \mathbf{C}(s) = [\mathbf{1} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\ \mathbf{G}(s)\mathbf{R}(s)$$

Provided that $\mathbf{1} + \mathbf{G}(s)\mathbf{H}(s)$ is nonsingular. The closed-loop transfer-matrix is defined as

$$\mathbf{T}(s) = [\mathbf{1} + \mathbf{G}(s)\mathbf{H}(s)]^{-1}\mathbf{G}(s)$$

$$\therefore \quad \mathbf{C}(s) = \mathbf{T}(s)\mathbf{R}(s) \tag{3.13}$$

3.1.3 Procedure for Drawing a Block Diagram

A control system can be represented diagrammatically by a block diagram. To draw a block diagram for a system, first write the differential equations that describe the dynamic behaviour of each component. Then take the laplace transforms of these differential equations, assuming zero initial conditions to obtain the algebraic equations. These equations will have variables and constants. From the working knowledge of the system, the input and output variables are identified and the block diagram for each equation can be drawn. Each equation gives one section of block diagram, the output of one section will be the input for another section. The various sections are interconnected to obtain the overall block diagram of the system.

As an example, consider the *RC* circuit shown in Figure 3.7(a). The equations for the circuit are as follows:

$$i = \frac{e_i - e_o}{R} \tag{3.14}$$

$$e_o = \frac{1}{C}\int i\,dt \tag{3.15}$$

The Laplace transform of Eqs. (3.14) and (3.15) with zero initial conditions yields

$$I(s) = \frac{E_i(s) - E_o(s)}{R} \tag{3.16}$$

$$E_o(s) = \frac{1}{Cs} I(s) \tag{3.17}$$

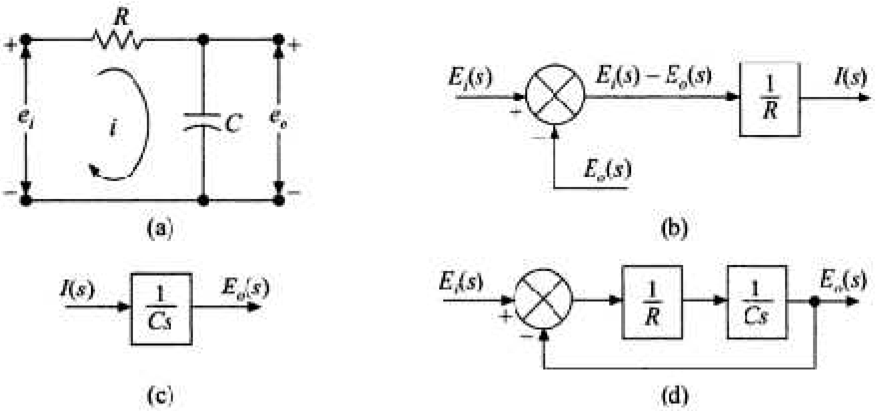


Figure 3.7 Block diagram of an electrical network.

Equation (3.16) represents a summing operation and the corresponding diagram is shown in Figure 3.7(b). Equation (3.17) represents the block as shown in Figure 3.7(c). Assembling these two elements, i.e. $I(s)$ and $E_o(s)$ we obtained the overall block diagram for the system as shown in Figure 3.7(d).

3.1.4 Block Diagram Reduction

It is important to note that blocks can be connected in series, if the output of one block is not affected by the next following block. If there are any loading effects between the components, it is necessary to combine these components into a single block. Any number of cascaded blocks representing non-loading components can be replaced by a single block, the transfer function of which is simply the product of the individual transfer functions.

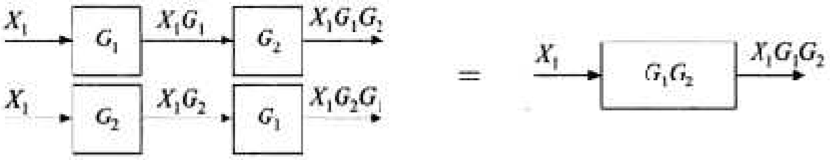
A complicated block diagram involving many feedback loops can be simplified by a step-by-step rearrangement, using the rules of block diagram algebra. Some of these important rules are given in Table 3.1. They are obtained by writing the same equations in a different way.

Simplification of the block diagram by rearrangements and substitutions considerably reduces the labour needed for subsequent mathematical analysis. It should be noted, however, that as the block diagram is simplified, the transfer functions in new blocks become more complex because new poles and zeros are generated.

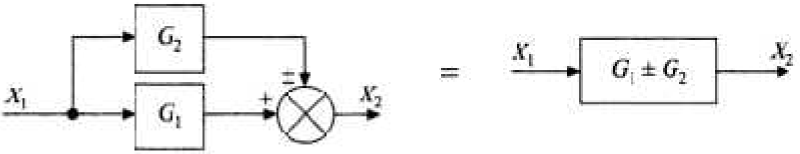
In simplifying a block diagram, the following points should be remembered.

1. The product of the transfer functions in the feed forward direction must remain the same.
2. The product of the transfer functions around the loop must remain the same.

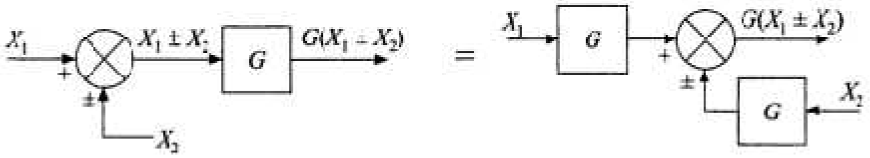
Table 3.1 Important rules for block diagram reduction



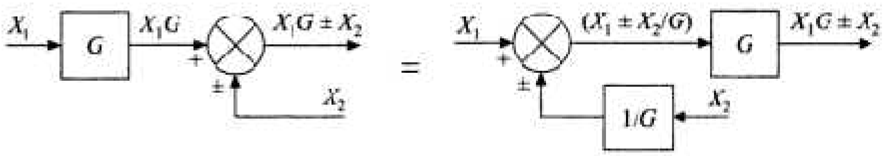
1. Combining blocks in cascade



2. Combining blocks in parallel

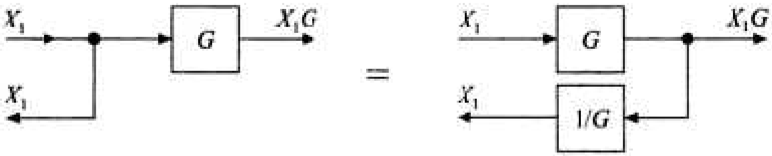


3. Moving a summing point after a block

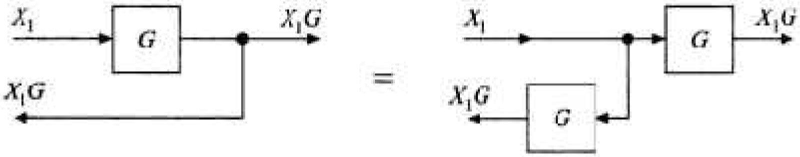


4. Moving a summing point ahead of a block

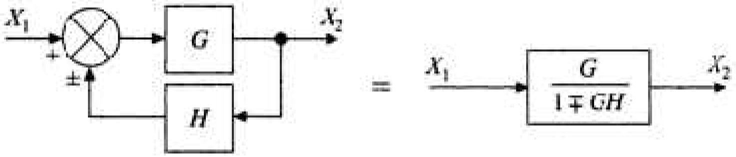
(Contd.)



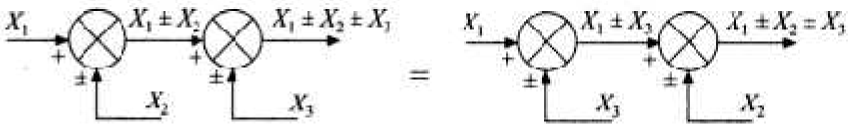
5. Moving a take-off point after a block



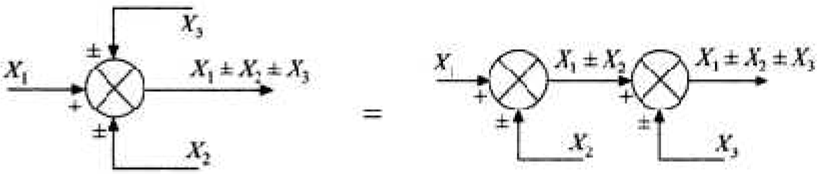
6. Moving a take off point ahead of a block



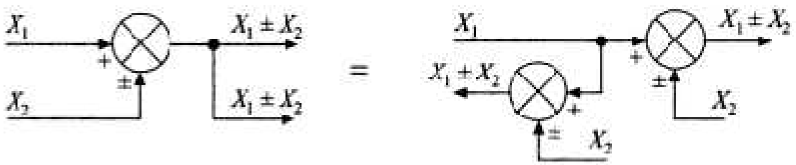
7. Eliminating a feedback loop



8. Interchanging summing points

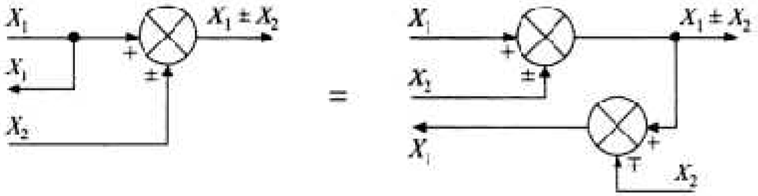


9. Splitting a summing point

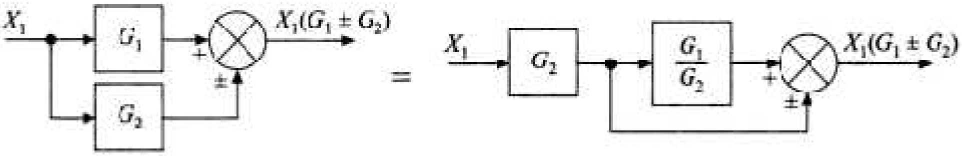


10. Moving a take-off point ahead of a summing point

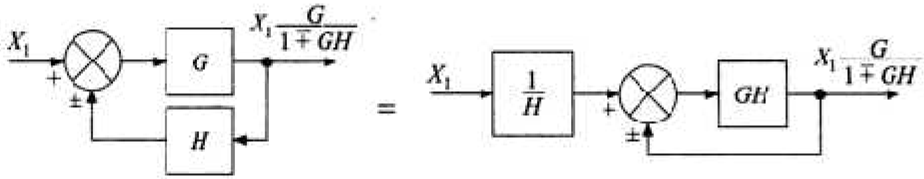
(Contd.)



11. Moving a take-off point after a summing point



12. Removing a block from a forward path



13. Removing a block from a feedback path

3.2 SIGNAL FLOW GRAPHS

A signal flow graph may be regarded as a simplified version of a block diagram. The usual application of signal flow graph is in system diagramming. Signal flow graphs were developed by S.J. Mason. Unlike the block diagram reduction process, they do not require any reduction process to obtain the transfer function of a complicated system, because of the availability of a flow graph gain formula which relates the input and output system variables. Mason's gain formula is especially useful in reducing large and complex system diagrams in one step, without requiring step-by-step reductions. A signal flow graph and a block diagram contain essentially the same information and one is in no sense superior to the other. Besides the difference in physical appearances of the signal flow graph and the block diagram, the signal flow graph may be regarded as constrained by more rigid mathematical rules than the block diagrams.

A signal flow graph is a graphical representation of the relationship between the variables of a set of linear algebraic equations written in the form of cause-and-effect relations. It consists of a network in which nodes representing each of the system variables are connected by directed branches. The directed branches have associated branch gains. A signal can be transmitted through a branch only in the direction of the arrow. Various terms used in the formulation of a signal flow graph are given below.

1. **Node:** A node (or junction point) represents a system variable which is equal to the sum of all the incoming signals at the node. Outgoing signals from the node do not affect the value of the node variable.
2. **Branch:** A branch is a line segment of the signal flow graph joining two nodes along which a signal travels from one node to the other in the direction indicated by the branch arrow and in the process gets multiplied by the gain or transmittance of the branch.
3. **Input node or Source:** An input node is a node with only outgoing branches. It does not have any incoming branches.
4. **Output node or Sink:** An output node is a node with only incoming branches. It does not have any outgoing branches. Any node can be converted into an output node by connecting an additional branch with unit transmittance to it.
5. **Mixed node:** A mixed node is a node that has both incoming and outgoing branches.
6. **Path:** A path is the traversal of connected branches in the direction of the branch arrows such that no node is traversed more than once.
7. **Forward path:** A forward path is a path that starts at an input node and ends at an output node, and along which no node is traversed more than once.
8. **Loop:** A loop is a path which originates and terminates at the same node and along which no node is traversed more than once.
9. **Nontouching loops:** Nontouching loops are loops which do not possess any common node.
10. **Path gain:** The product of the branch gains encountered in traversing a path is called the path gain.
11. **Forward path gain:** The product of the branch gains encountered in traversing a forward path is called the forward path gain.
12. **Loop gain:** The product of the branch gains encountered in traversing the loop is called the loop gain.
13. **Self loop:** A self loop is a loop consisting of a single branch.

3.2.1 Construction of Signal Flow Graphs

The signal flow graph of a system is constructed from its describing equations. In general, given a set of algebraic equations, the construction of the signal flow graph is basically a matter of following through the cause-and-effect relations relating each variable in terms of itself and others. To outline the procedure, let us consider a system described by the following set of equations.

$$x_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$$

$$x_3 = a_{23}x_2$$

$$x_4 = a_{34}x_3 + a_{44}x_4$$

$$x_5 = a_{35}x_3 + a_{45}x_4$$

where x_1 is the input variable and x_5 is the output variable.

The signal flow graph for this system is constructed as shown in Figure 3.8. First, the nodes are located as shown in Figure 3.8(a). The first equation states that x_2 is equal to the sum of five signals and its signal flow graph is shown in Figure 3.8(b). Similarly, the signal flow graphs for the remaining three equations are constructed as shown in Figure 3.8(c), Figures 3.8(d) and 3.8(e) respectively giving the complete signal flow graph of Figure 3.8(f).

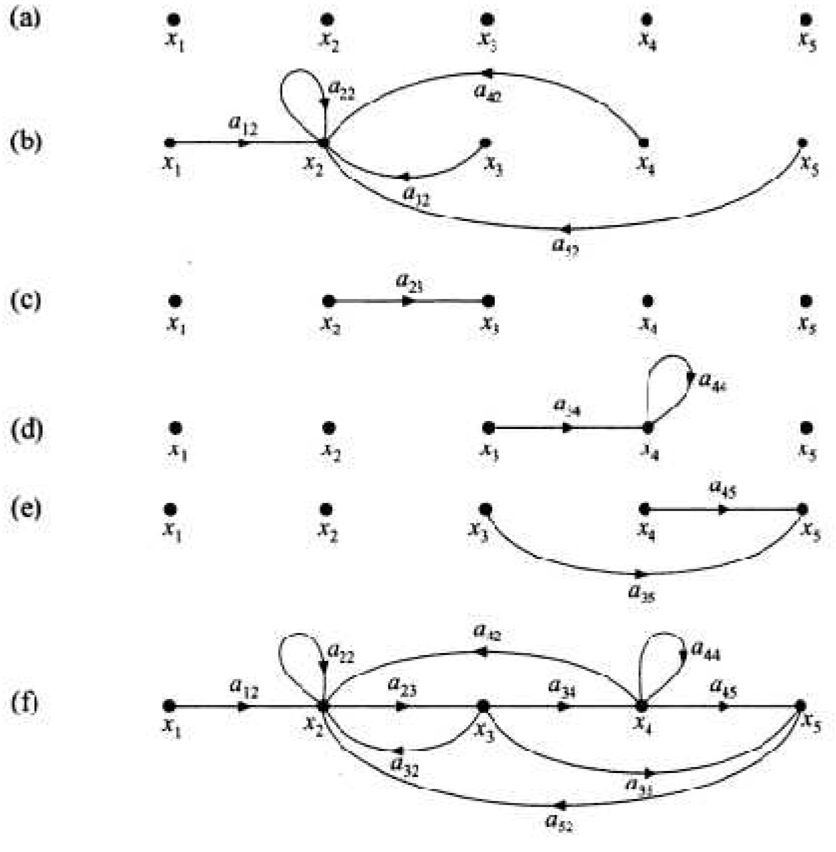


Figure 3.8 Signal flow graph.

The overall gain from input to output may be obtained by applying Mason's gain formula.

The relationship between an input variable and an output variable of a signal flow graph is given by the net gain between the input and output nodes and is known as the overall gain of the system. Mason's gain formula for the determination of the overall system gain is given by

$$T = \frac{\Sigma_k M_k \Delta_k}{\Delta} \tag{3.18}$$

where M_k = path gain of the kth forward path

Δ_k = value of Δ for that part of the graph not touching the kth forward path

T = Overall gain of the system

Δ = determinant of the signal flow graph
= 1 – (sum of the loop gains of all individual loops)
+ (sum of the gain products of all possible combinations of two nontouching loops)
– (sum of the gain products of all possible combinations of three nontouching loops)
+ ⋯, i.e.

$$\Delta = 1 - \Sigma_p M_{p1} + \Sigma_p M_{p2} - \Sigma_p M_{p3} + \cdots \tag{3.19}$$

where M_{pr} is the gain product of pth possible combination of r nontouching loops.

Let us illustrate the use of Mason's gain formula by finding the overall gain of the signal flow graph shown in Figure 3.8(f). The following conclusions are drawn by inspection of this signal flow graph. The details are shown in Figure 3.9.

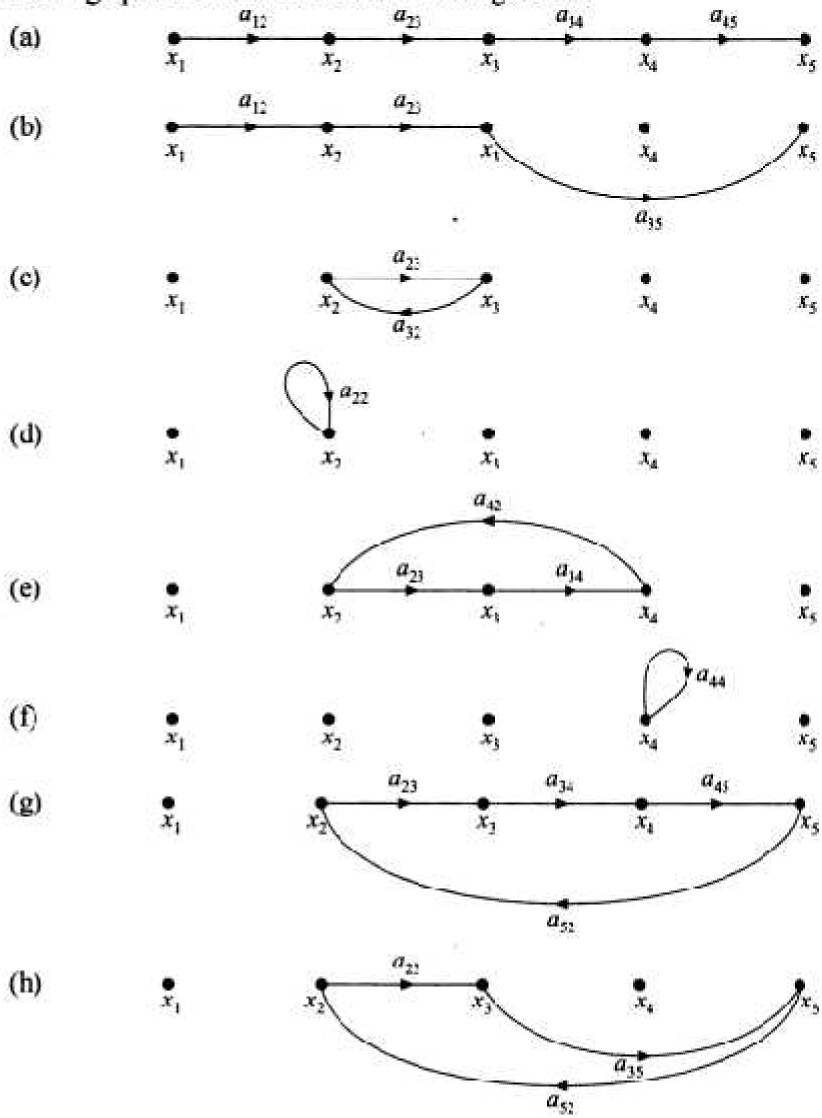


Figure 3.9 (Contd.)

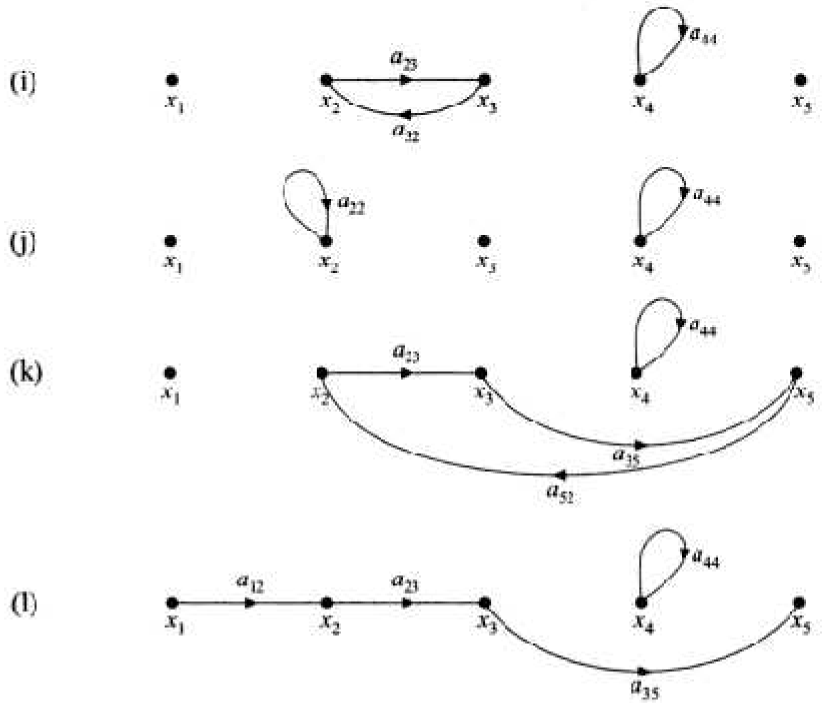


Figure 3.9 Use of Mason's gain formula.

1. There are two forward paths shown in Figure 3.9(a) and (b) with path gains:

   Forward path x_1-x_2-x_3-x_4-x_5 $\quad M_1 = a_{12}\, a_{23}\, a_{34}\, a_{45}$

   Forward path x_1-x_2-x_3-x_5 $\quad M_2 = a_{12}\, a_{23}\, a_{35}$

2. There are six individual loops shown in Figure 3.9(c), (d), (e), (f), (g) and (h) with loop gains:

   Loop x_2-x_3-x_2 $\quad L_1 = a_{23}\, a_{32}$

   Loop x_2-x_2 $\quad L_2 = a_{22}$

   Loop x_2-x_3-x_4-x_2 $\quad L_3 = a_{23}\, a_{34}\, a_{42}$

   Loop x_4-x_4 $\quad L_4 = a_{44}$

   Loop x_2-x_3-x_4-x_5-x_2 $\quad L_5 = a_{23}\, a_{34}\, a_{45}\, a_{52}$

   Loop x_2-x_3-x_5-x_2 $\quad L_6 = a_{23}\, a_{35}\, a_{52}$

3. There are three possible combinations of two nontouching loops shown in Figure 3.9(i), (j) and (k) with loop gain products:

   Loops x_2-x_3-x_2 and x_4-x_4 $\quad L_{14} = a_{23}\, a_{32}\, a_{44}$

   Loops x_2-x_2 and x_4-x_4 $\quad L_{24} = a_{22}\, a_{44}$

   Loops x_2-x_3-x_5-x_2 and x_4-x_4 $\quad L_{64} = a_{23}\, a_{35}\, a_{52}\, a_{44}$

4. There are no combinations of three nontouching loops, four nontouching loops, etc. Therefore,

$$M_{p3} = M_{p4} = \cdots = 0$$

Hence from Eq. (3.19) for Δ

$$\Delta = 1 - (a_{23}\,a_{32} + a_{22} + a_{23}\,a_{34}\,a_{42} + a_{44} + a_{23}\,a_{34}\,a_{45}\,a_{52} + a_{23}\,a_{35}\,a_{52})$$
$$+ (a_{23}\,a_{32}\,a_{44} + a_{22}\,a_{44} + a_{23}\,a_{35}\,a_{52}\,a_{44})$$

5. The first forward path is touching all the loops. Therefore, $\Delta_1 = 1$. The second forward path is not touching one loop (with gain a_{44}) shown in Figure 3.9(l). Therefore, $\Delta_2 = 1 - a_{44}$.

   Using the Mason's gain formula, the overall gain is

$$T = \frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$= \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}(1 - a_{44})}{1 - a_{23}a_{32} - a_{22} - a_{23}a_{34}a_{42} - a_{44} - a_{23}a_{34}a_{45}a_{52} - a_{23}a_{35}a_{52} + a_{23}a_{32}a_{44} + a_{22}a_{44} + a_{23}a_{35}a_{52}a_{44}}$$

3.2.2 Basic Properties of Signal Flow Graph

1. Signal flow graph applies only to linear systems.
2. The equations for which a signal flow graph is drawn must be algebraic equations in the form of cause-and-effect.
3. Nodes are used to represent variables. Normally the nodes are arranged from left to right, from the input to the output following a succession of cause-and-effect relations through the system.
4. Signals travel along branches only in the direction described by the arrows of the branches.
5. The branch directing from node x_k to x_j represents the dependence of x_j upon x_k but not the reverse.
6. A signal x_k traveling along a branch between x_k and x_j is multiplied by the gain of the branch a_{kj} so that a signal $a_{kj}\,x_k$ is delivered at x_j.
7. For a given system, a signal flow graph is not unique. Many different signal flow graphs can be drawn for a given system by writing the system equations differently.

3.2.3 Signal Flow Graph Algebra

Based on the properties of the signal flow graph, we can outline the following manipulation rules and algebra.

1. The value of the variable represented by a node is equal to the sum of all the signals entering the node. For the signal flow graph of Figure 3.10, the value of y_1 is equal to the sum of the signals transmitted through all the incoming branches; i.e.

$$y_1 = a_{21}y_2 + a_{31}y_3 + a_{41}y_4 + a_{51}y_5$$

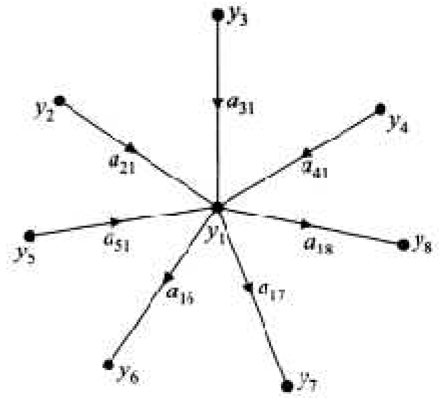


Figure 3.10 Node as a summing point and transmitting point.

2. The value of the variable represented by a node is transmitted through all branches leaving the node. In the signal flow graph of Figure 3.10, we have

$$y_6 = a_{16}y_1$$
$$y_7 = a_{17}y_1$$
$$y_8 = a_{18}y_1$$

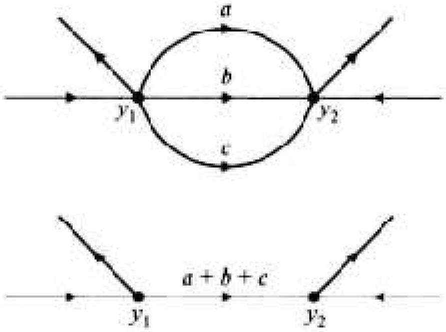


Figure 3.11 Signal flow graph with parallel paths replaced by one with a single branch.

3. Parallel branches in the same direction connecting two nodes can be replaced by a single branch with gain equal to the sum of the gains of the parallel branches. An example of this case illustrated in Figure 3.11

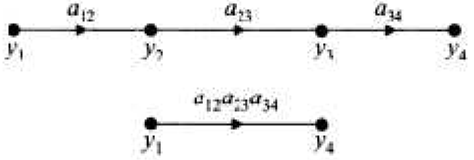


Figure 3.12 Signal flow graph with cascade unidirectional branches replaced by a single branch.

4. A series connection of unidirectional branches as shown in Figure 3.12 can be replaced by a single branch with gain equal to the product of the branch gains.

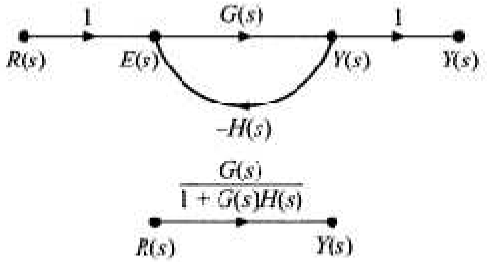


Figure 3.13 Signal flow graph of the single loop feedback control system replaced by a single branch.

5. A single feedback loop can be replaced by a single branch with gain as shown in Figure 3.13.

3.2.4 Construction of Signal Flow Graph for Control Systems

A control system can be represented diagrammatically by a signal flow graph. The control system is usually described in time-domain by differential equations. These differential equations are first converted into algebraic equations using the Laplace transform, and these equations are written in cause-and-effect form. The constants and the variables of the equation are identified. From the working knowledge of the system, the variables are identified as input, output and intermediate variables. For each variable, a node is assigned in the signal flow graph and the constants are the gain or transmittance of the branches connecting the nodes. A signal flow graph is drawn for each equation and all those signal flow graphs are interconnected to obtain the overall signal flow graph of the system.

The block diagram and the signal flow graph of a system provide essentially the same information, and one is in no way superior to the other but there is no standard procedure for reducing the block diagram to obtain the transfer function of the system. Also the block diagram reduction process is tedious and time consuming as it is not systematic, and it is difficult to choose the rule to be applied for simplification. It mainly depends on one's knowledge and experience. On the other hand, there is a standard formula called Mason's gain formula using which the transfer function of a system can be obtained in one step from the signal flow of a system. So, block diagram is to be converted into signal flow graph.

The following procedure can be followed to convert a block diagram into a signal flow graph.

1. In the given block diagram, assume nodes at input, output, at every summing point, at every branch point and in between cascade blocks.
2. Draw the nodes separately as big thick dots and number the dots in the order 1, 2, 3, ...
3. From the block diagram, find the gain between the nodes in the main forward path and connect all the corresponding nodes by directed straight line segments and mark the gain between the nodes on the segment.

4. Draw the feed forward paths between various nodes and mark the gain between nodes on the directed branches.
5. Draw the feedback paths between various nodes and mark the gain of feedback paths along with the sign.

A comparison of a block diagram method and a signal flow method is given in Table 3.2.

Table 3.2 Comparison of block diagram and signal flow graph methods

Block diagram method	*Signal flow graph method*
1. It is a pictorial representation of the functions performed by each component and of the flow of signals.	1. It is a graphical representation of a relationship between variables of a set of linear algebraic equations written in the form of cause-and-effect relations.
2. It can be used to represent linear as well as nonlinear systems.	2. It can be used to represent only linear systems.
3. No direct formula is available to find the overall transfer function of the system.	3. Mason's gain formula is available to find the overall transfer function of the system.
4. Step-by-step procedure is to be followed to find the transfer function.	4. Transfer function can be obtained in one step.
5. It is not a systematic method.	5. It is a systematic method.
6. It indicates more realistically the signal flows of the system than the original system itself.	6. It is constrained by more rigid mathematical rules than a block diagram.
7. For a given system, the block diagram is not unique. Many dissimilar and unrelated systems can be represented by the same block diagram.	7. For a given system, the signal flow graph is not unique.

Example 3.1 Obtain the transfer function of the control system whose block diagram is shown in Figure 3.14 by (a) block diagram reduction technique and (b) signal flow graph method.

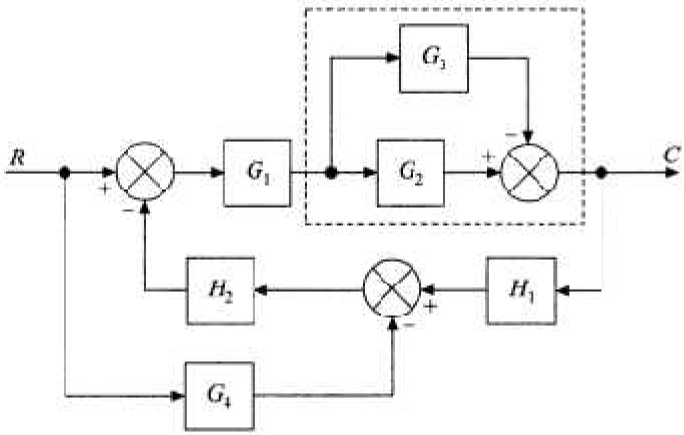


Figure 3.14 Example 3.1.

Solution: (a) *Block diagram reduction technique*

1. In the block diagram shown in Figure 3.14, blocks G_2 and G_3 are in parallel. They can be combined into a single block with a gain of $(G_2 - G_3)$ as shown in Figure 3.15.

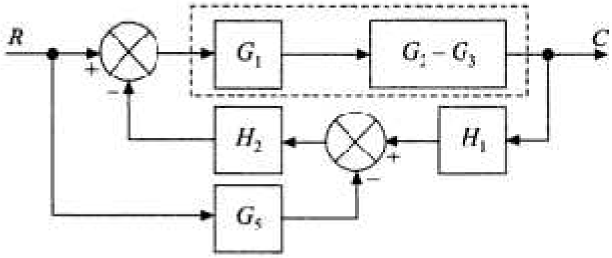


Figure 3.15

2. In Figure 3.15, blocks G_1 and $(G_2 - G_3)$ are in cascade. They can be combined into a single block with a gain of $G_1(G_2 - G_3)$ as shown in Figure 3.16.

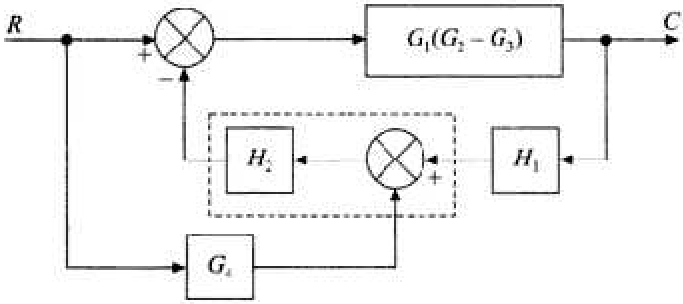


Figure 3.16

3. In Figure 3.16, move the summing point after the block H_2. The resultant block diagram is shown in Figure 3.17.

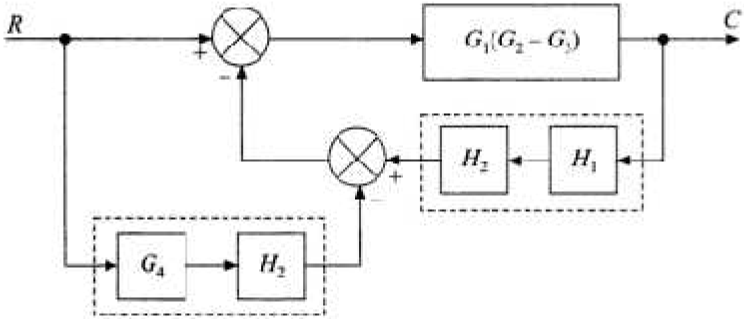


Figure 3.17

4. In Figure 3.17, blocks G_4 and H_2 are in cascade. They can be combined into a single block with a gain of $G_4\ H_2$. Also blocks H_1 and H_2 are in cascade. They can be combined into a single block with a gain of $H_1\ H_2$ as shown in Figure 3.18.

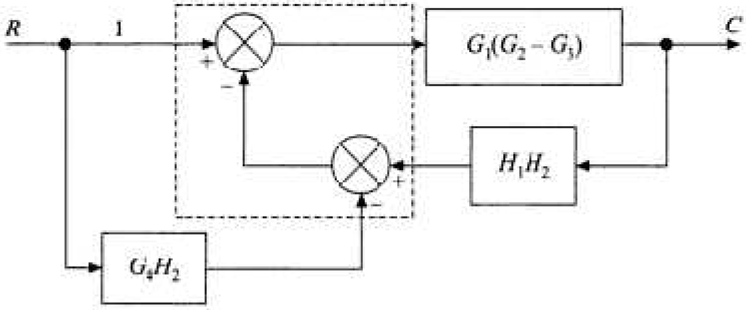


Figure 3.18

5. Interchanging the summing points in Figure 3.18, the block diagram is as shown in Figure 3.19.

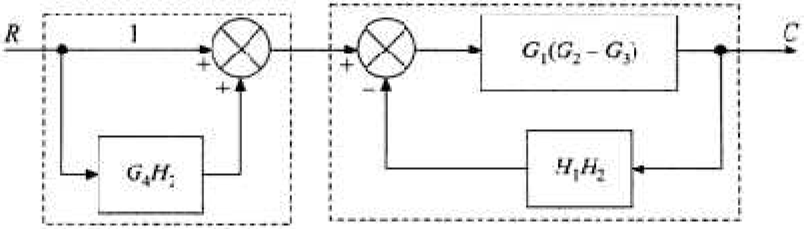


Figure 3.19

6. In Figure 3.19, combining 1 and G_4H_2 into a block with gain of $(1 + G_4H_2)$ and simplifying the feedback arrangement into a single block, the resultant block diagram is as shown in Figure 3.20.

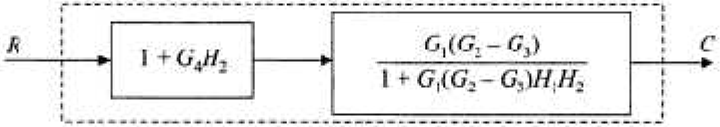


Figure 3.20

7. In Figure 3.20, combining the two blocks in cascade into a single block, the resultant block diagram is as shown in Figure 3.21.

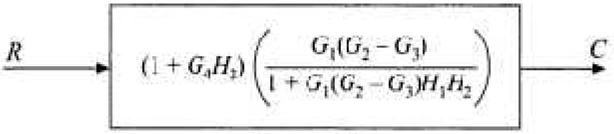


Figure 3.21

From Figure 3.21, the overall transfer function is

$$\frac{C}{R} = \frac{G_1G_2 - G_1G_3 + G_1G_2G_4H_2 - G_1G_3G_4H_2}{1 + G_1G_2H_1H_2 - G_1G_3H_1H_2}$$

(a) *Signal flow graph method*

The signal flow graph corresponding to the block diagram of Figure 3.14 is shown in Figure 3.22.

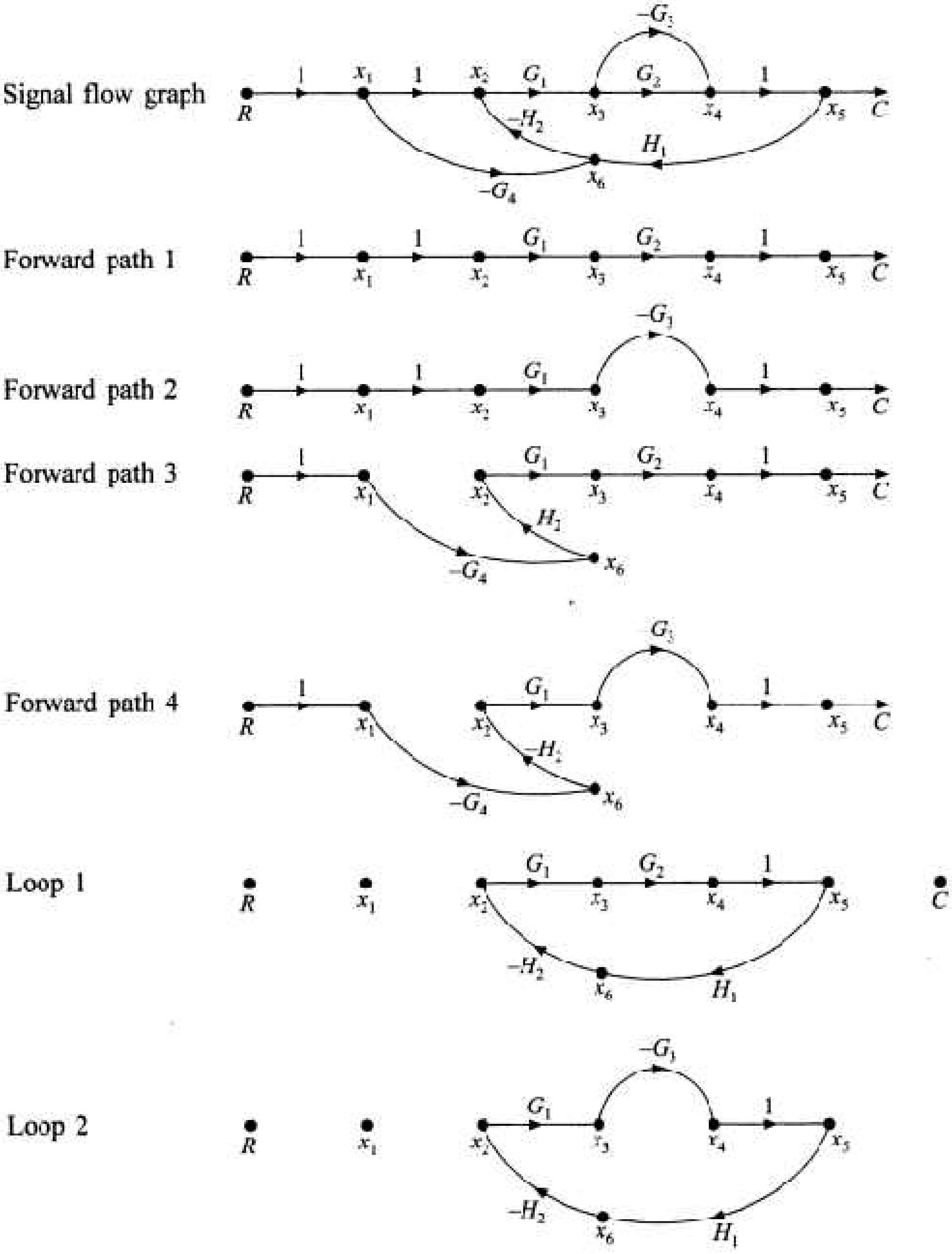


Figure 3.22

The signal flow graph shown in Figure 3.22 has four forward paths and two loops. There are no combinations of two nontouching loops. Both the loops are touching all the four forward paths.

The forward paths, the gains of those forward paths and the values of Δ associated with those paths are given as follows:

Forward path R-x_1-x_2-x_3-x_4-x_5-C

$$M_1 = (1)(1)(G_1)(G_2)(1) = G_1G_2 \qquad \Delta_1 = 1$$

Forward path R-x_1-x_2-x_3-x_4-x_5-C

$$M_2 = (1)(1)(G_1)(-G_3)(1) = -G_1G_3 \qquad \Delta_2 = 1$$

Forward path R-x_1-x_6-x_2-x_3-x_4-x_5-C

$$M_3 = (1)(-G_4)(-H_2)(G_1)(G_2)(1) = G_1G_2G_4H_2 \qquad \Delta_3 = 1$$

Forward path R-x_1-x_6-x_2-x_3-x_4-x_5-C

$$M_4 = (1)(-G_4)(-H_2)(G_1)(-G_3)(1) = -G_1G_3G_4H_2 \qquad \Delta_4 = 1$$

Loops and the gains associated with them are as follows:

Loop x_2-x_3-x_4-x_5-x_6-x_2 $\qquad L_1 = (G_1)(G_2)(H_1)(-H_2) = -G_1G_2H_1H_2$

Loop x_2-x_3-x_4-x_5-x_6-x_2 $\qquad L_2 = (G_1)(-G_3)(1)(H_1)(-H_2) = G_1G_3H_1H_2$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2)$

$$\therefore \qquad \Delta = 1 - (-G_1G_2H_1H_2 + G_1G_3H_1H_2) = 1 + G_1G_2H_1H_2 - G_1G_3H_1H_2$$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 + M_4\Delta_4}{\Delta}$$

$$= \frac{G_1G_2 - G_1G_3 + G_1G_2G_4H_2 - G_1G_3G_4H_2}{1 + G_1G_2H_1H_2 - G_1G_3H_1H_2}$$

Example 3.2 Obtain the transfer function of the feedback control system shown in Figure 3.23 by (a) block diagram reduction technique and (b) signal flow graph method.

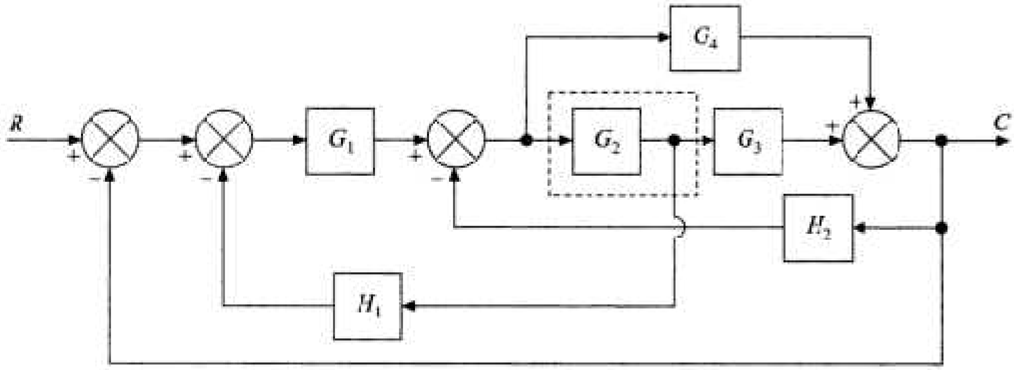


Figure 3.23 Example 3.2.

Solution: (a) *Block diagram reduction method*

1. Moving the take-off point ahead of block G_2, in Figure 3.23, the block diagram will be as shown in Figure 3.24.

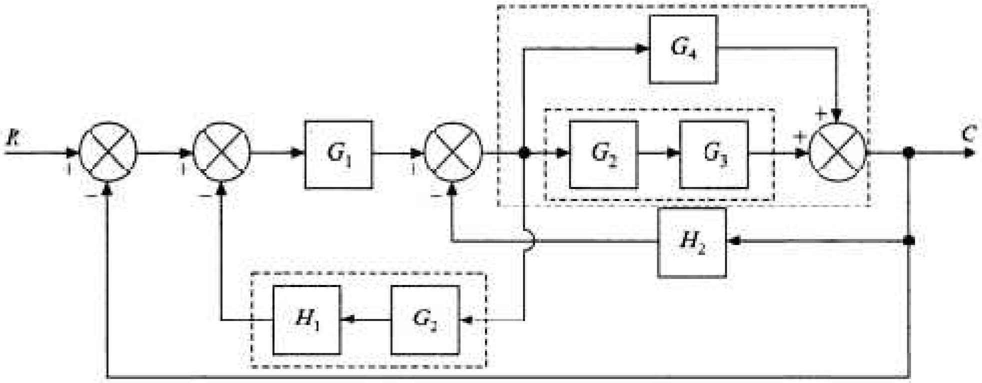


Figure 3.24

2. In Figure 3.24, blocks G_2 and G_3 are in cascade. They can be combined into a single block with a gain of G_2G_3. Now blocks (G_2G_3) and G_4 are in parallel. They can be combined into a single block with a gain of $(G_2G_3 + G_4)$. Also blocks G_2 and H_1 are in cascade. They can be combined into a single block with a gain of G_2H_1. The resultant block diagram is as shown in Figure 3.25.

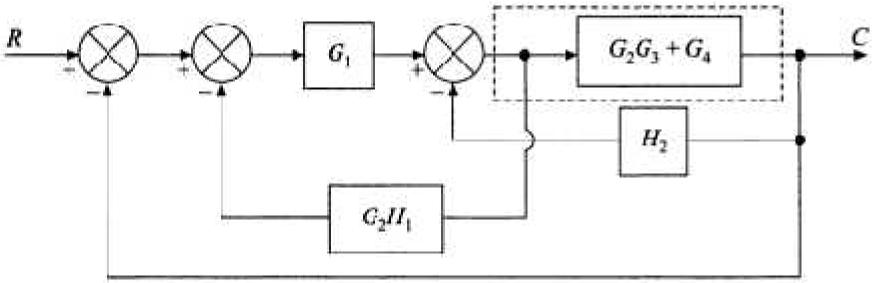


Figure 3.25

3. In Figure 3.25, moving the take-off point after the block $(G_2G_3 + G_4)$, the block diagram will be as shown in Figure 3.26.

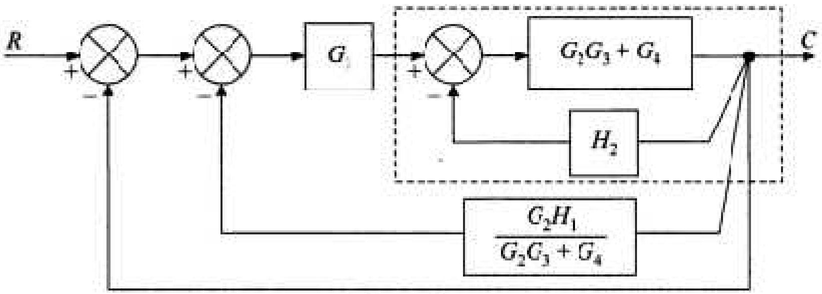


Figure 3.26

4. Simplifying the inner loop of Figure 3.26, the block diagram is as shown in Figure 3.27.

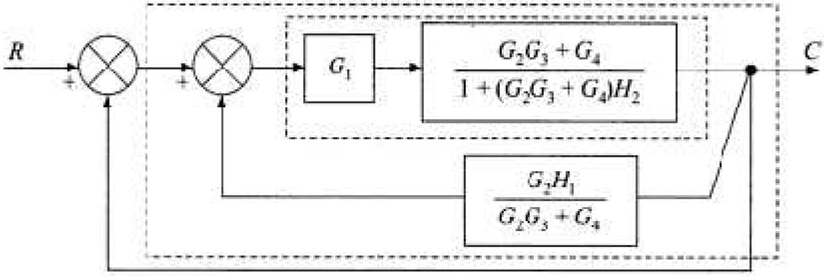


Figure 3.27

5. In Figure 3.27, combining the two blocks in cascade in the forward path and then simplifying the inner loop, the block diagram is as shown in Figure 3.28.

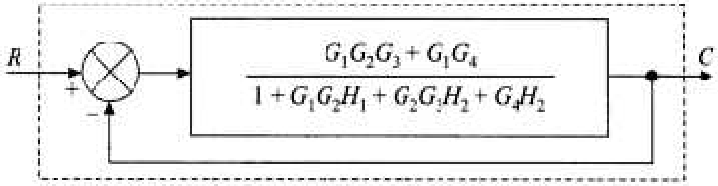


Figure 3.28

6. Simplifying the single loop in Figure 3.28, the block diagram is as shown in Figure 3.29.

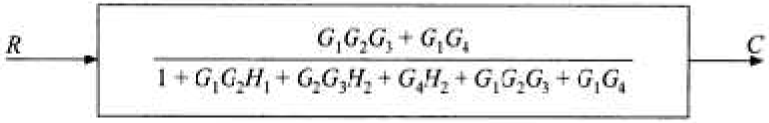


Figure 3.29

From Figure 3.29, the transfer function is therefore

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1+G_1G_2H_1+G_2G_3H_2+G_4H_2+G_1G_2G_3+G_1G_4}$$

(b) *Signal flow graph method*

The signal flow graph for the system with the block diagram of Figure 3.23 is shown in Figure 3.30.

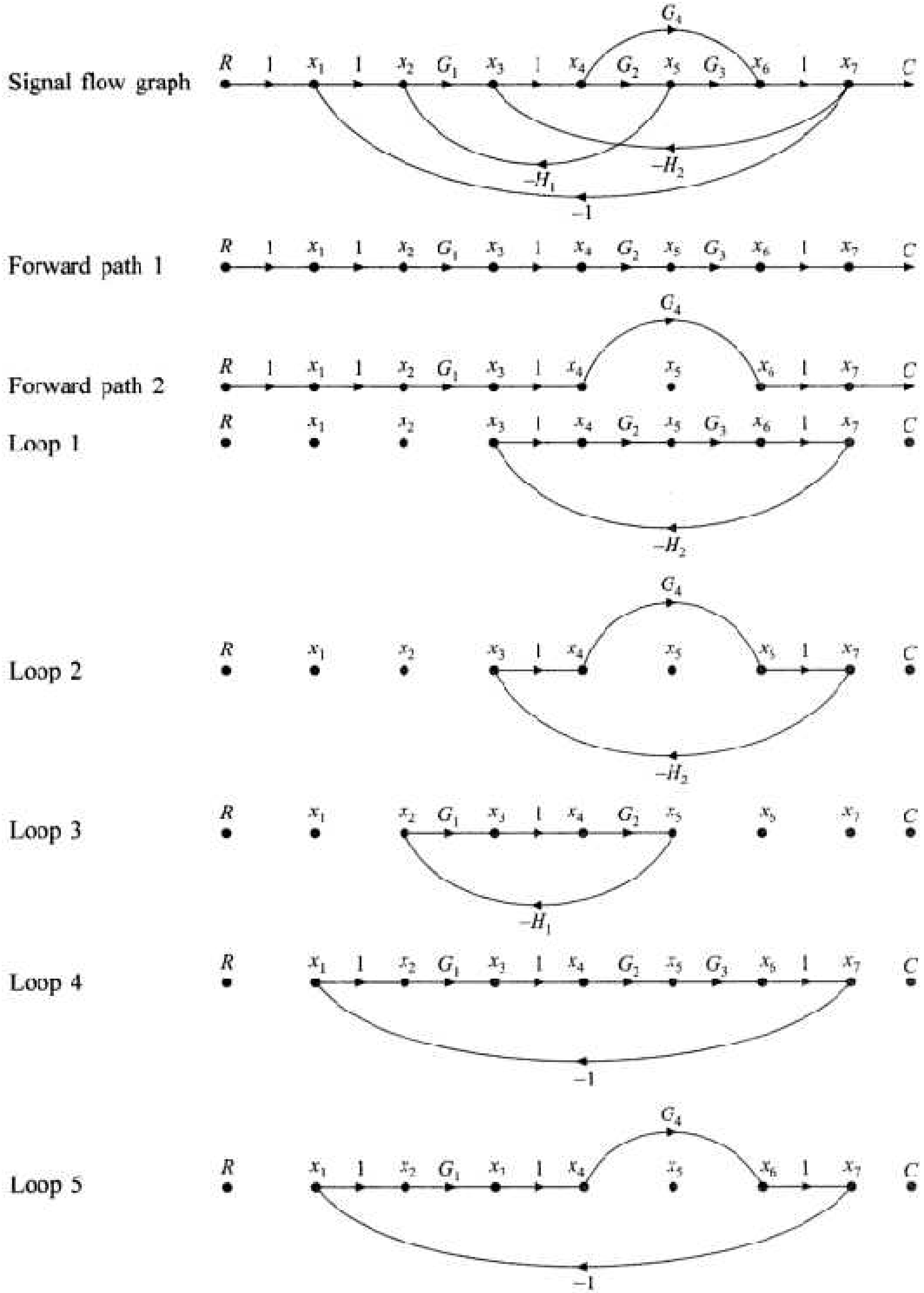

Signal flow graph
R 1 x_1 1 x_2 G_1 x_3 1 x_4 G_2 x_5 G_3 x_6 1 x_7 C
G_4
$-H_1$
$-H_2$
-1
Forward path 1
Forward path 2
Loop 1
Loop 2
Loop 3
Loop 4
Loop 5

Figure 3.30

In the signal flow graph shown in Figure 3.30, there are two forward paths and five loops as shown and there are no two nontouching loops and all the loops are touching both the forward paths.

The forward paths and the gains associated with them are given as follows:

Forward path R-x_1-x_2-x_3-x_4-x_5-x_6-x_7-C

$$M_1 = (1)(1)(G_1)(1)(G_2)(G_3)(1) = G_1G_2G_3 \qquad \Delta_1 = 1$$

Forward path R-x_1-x_2-x_3-x_4-x_6-x_7-C

$$M_2 = (1)(1)(G_1)(1)(G_4)(1) = G_1G_4 \qquad \Delta_2 = 1$$

The loops and the gains associated with them are given as follows:

Loop x_3-x_4-x_5-x_6-x_7-x_3 $\quad L_1 = (1)(G_2)(G_3)(1)(-H_2) = -G_2G_3H_2$

Loop x_3-x_4-x_6-x_7-x_3 $\quad L_2 = (1)(G_4)(1)(-H_2) = -G_4H_2$

Loop x_2-x_3-x_4-x_5-x_2 $\quad L_3 = (G_1)(1)(G_2)(-H_1) = -G_1G_2H_1$

Loop x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_1 $\quad L_4 = (1)(G_1)(1)(G_2)(G_3)(1)(-1) = -G_1G_2G_3$

Loop x_1-x_2-x_3-x_4-x_6-x_7-x_1 $\quad L_5 = (1)(G_1)(1)(G_4)(1)(-1) = -G_1G_4$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$$

$$\Delta = 1 - (-G_2G_3H_2 - G_4H_2 - G_1G_2H_1 - G_1G_2G_3 - G_1G_4)$$
$$= 1 + G_2G_3H_2 + G_4H_2 + G_1G_2H_1 + G_1G_2G_3 + G_1G_4$$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_2G_3H_2 + G_4H_2 + G_1G_2H_1 + G_1G_2G_3 + G_1G_4}$$

Example 3.3 Simplify the block diagram shown in Figure 3.31 and obtain the closed-loop transfer function $C(s)/R(s)$. Verify the result by the signal flow graph method.

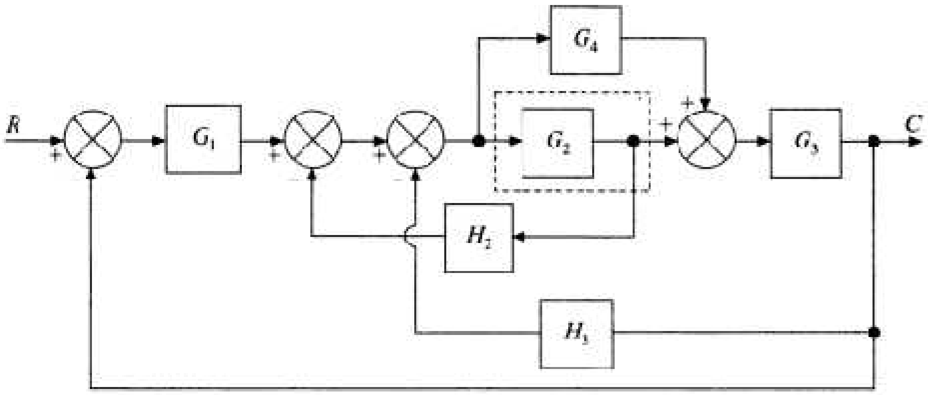


Figure 3.31 Example 3.3.

Solution: (a) *Block diagram reduction technique*

1. In Figure 3.31, moving the take-off point ahead of the block G_2, the block diagram will be as shown in Figure 3.32.

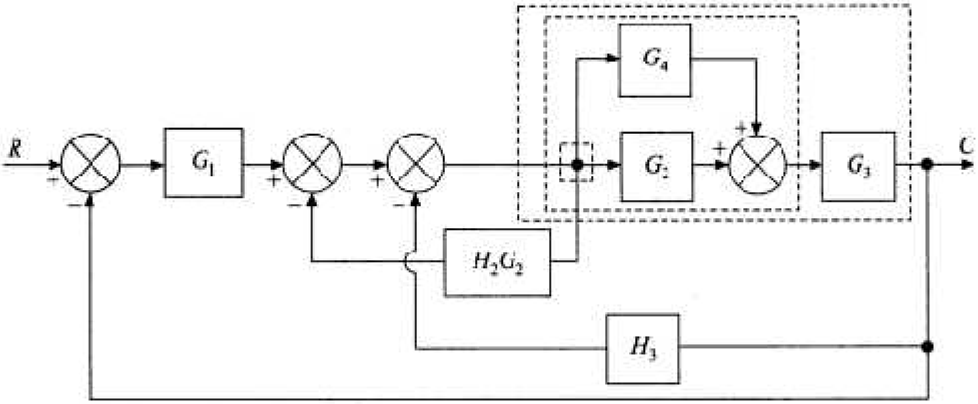


Figure 3.32

2. In Figure 3.32, blocks G_2 and G_4 are in parallel. They can be combined into a single block with a gain of $(G_2 + G_4)$. Now the blocks $(G_2 + G_4)$ and G_3 are in cascade. They can be combined into a single block with a gain of $G_3(G_2 + G_4)$. Moving the take-off point after the block $G_3(G_2 + G_4)$, the block diagram will be as shown in Figure 3.33.

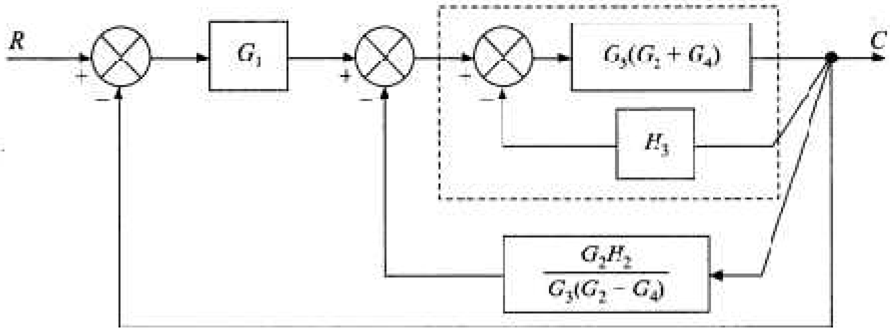


Figure 3.33

3. Reducing the inner loop in Figure 3.33, the block diagram reduces to that shown in Figure 3.34.

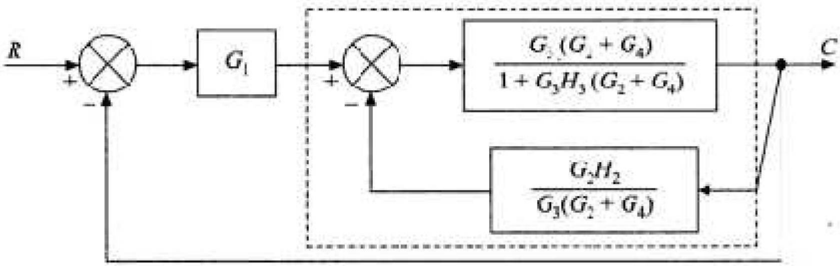


Figure 3.34

4. Reducing the inner loop in Figure 3.34, the block diagram reduces to that shown in Figure 3.35.

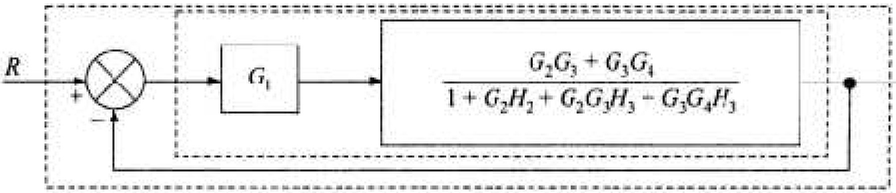


Figure 3.35

5. The two blocks in the forward path of Figure 3.35 are in cascade and can be combined into a single block. Simplifying the loop, the block diagram reduces to that shown in Figure 3.36.

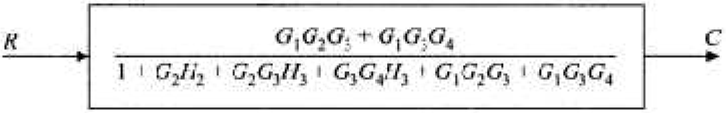


Figure 3.36

So, the transfer function of the system is

$$\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_3G_4}{1 + G_2H_2 + G_2G_3H_3 + G_3G_4H_3 + G_1G_2G_3 + G_1G_3G_4}$$

(b) *Signal flow graph method*

The signal flow graph corresponding to the block diagram of Figure 3.31 for the system is shown in Figure 3.37.

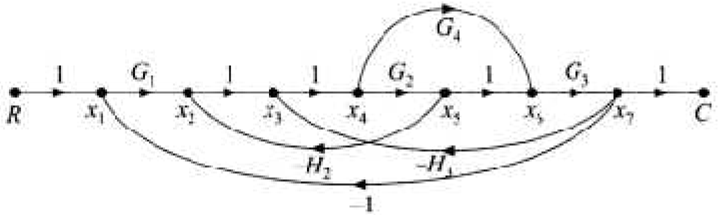


Figure 3.37

The signal flow graph in Figure 3.37 has two forward paths and five loops. There are no pairs of two nontouching loops and all the loops are touching both the forward paths.

The forward paths and gains associated with them are as follows:

Forward path $R\text{-}x_1\text{-}x_2\text{-}x_3\text{-}x_4\text{-}x_5\text{-}x_6\text{-}x_7\text{-}C$

$$M_1 = (1)(G_1)(1)(1)(G_2)(1)(G_3)(1) = G_1G_2G_3$$

Forward path R-x_1-x_2-x_3-x_4-x_6-x_7-C

$$M_2 = (1)(G_1)(1)(1)(G_4)(G_3)(1) = G_1G_3G_4$$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_4-x_5-x_2 $\quad L_1 = (1)(1)(G_2)(-H_2) = -G_2H_2$

Loop x_3-x_4-x_5-x_6-x_7-x_3 $\quad L_2 = (1)(G_2)(1)(G_3)(-H_3) = -G_2G_3H_3$

Loop x_3-x_4-x_6-x_7-x_3 $\quad L_3 = (1)(G_4)(G_3)(-H_3) = -G_3G_4H_3$

Loop x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_1 $\quad L_4 = (G_1)(1)(1)(G_2)(1)(G_3)(-1) = -G_1G_2G_3$

Loop x_1-x_2-x_3-x_4-x_6-x_7-x_1 $\quad L_5 = (G_1)(1)(1)(G_4)(G_3)(-1) = -G_1G_3G_4$

Since all the loops are touching both the forward paths, $\Delta_1 = 1$ and $\Delta_2 = 1$.
The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$

$$\therefore \quad \Delta = 1 - (-G_2H_2 - G_2G_3H_3 - G_3G_4H_3 - G_1G_2G_3 - G_1G_3G_4)$$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_1G_3G_4}{1 + G_2H_2 + G_2G_3H_3 + G_3G_4H_3 + G_1G_2G_3 + G_1G_3G_4}$$

Example 3.4 Obtain the transfer function of the control system represented by the block diagram shown in Figure 3.38, by the signal flow graph method.

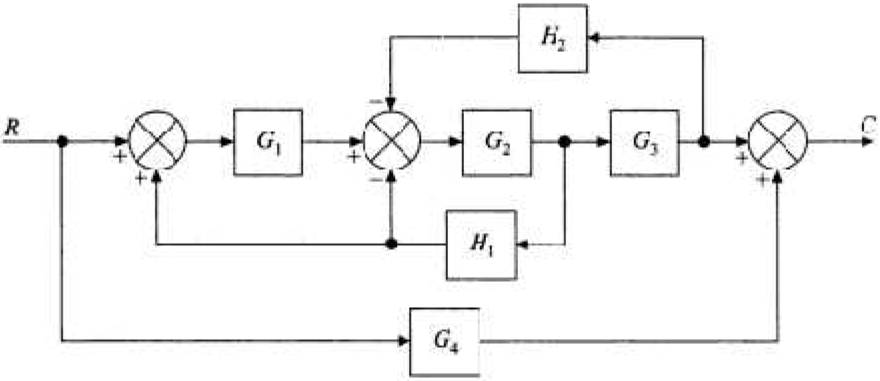


Figure 3.38 Example 3.4.

Solution: *Signal flow graph method*

The signal flow graph of the system obtained from the block diagram of Figure 3.38 is shown in Figure 3.39.

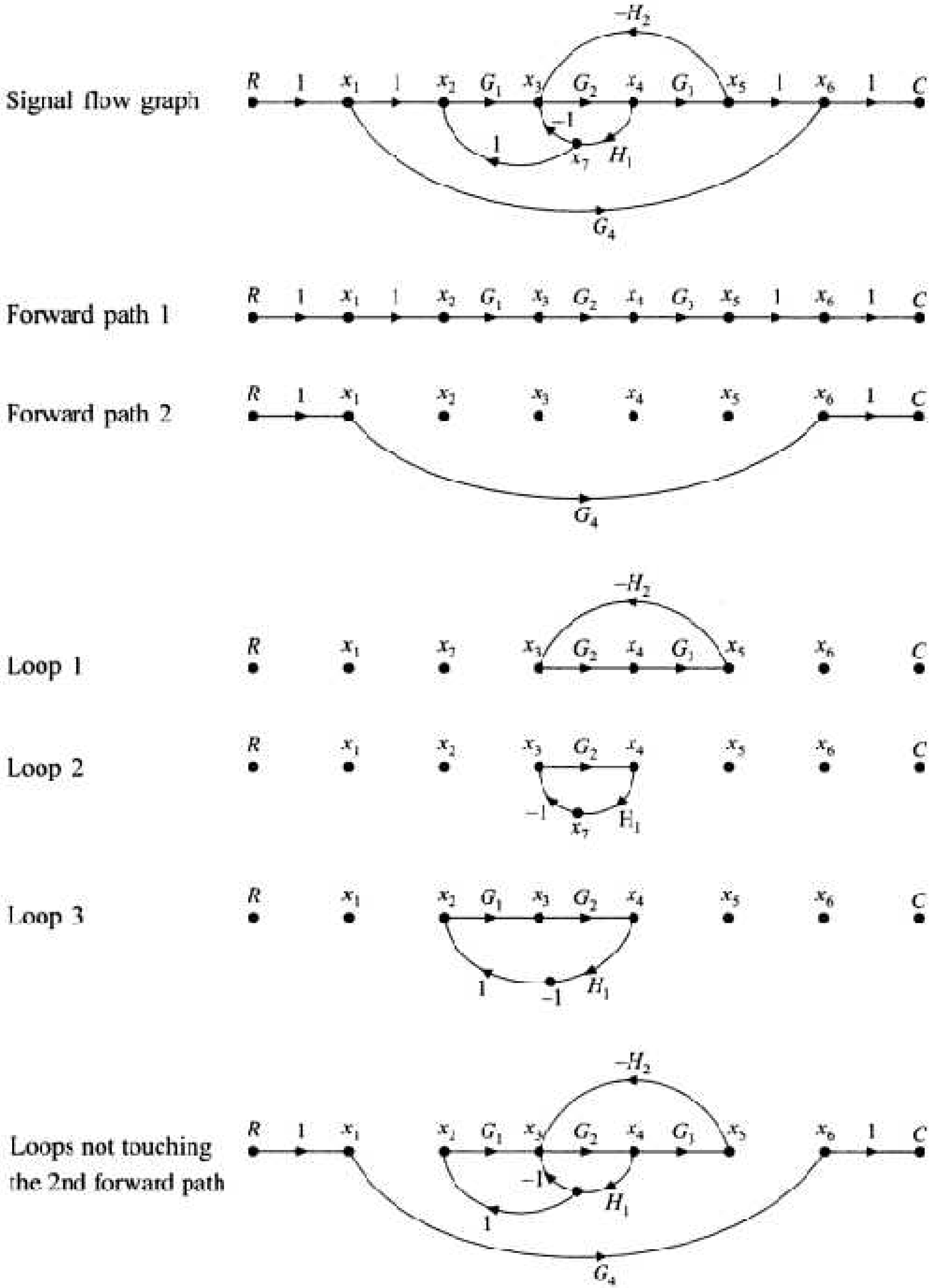

Signal flow graph
R 1 x1 1 x2 G1 x3 G2 x4 G3 x5 1 x6 1 C
−H2
−1
1
x7 H1
G4
Forward path 1
R 1 x1 1 x2 G1 x3 G2 x4 G3 x5 1 x6 1 C
Forward path 2
R 1 x1 x2 x3 x4 x5 x6 1 C
G4
Loop 1
R x1 x2 x3 G2 x4 G3 x5 x6 C
−H2
Loop 2
R x1 x2 x3 G2 x4 x5 x6 C
−1 x7 H1
Loop 3
R x1 x2 G1 x3 G2 x4 x5 x6 C
1 −1 H1
Loops not touching
the 2nd forward path
R 1 x1 x2 G1 x3 G2 x4 G3 x5 x6 1 C
−H2
−1 H1
1
G4

Figure 3.39

In the signal flow graph shown in Figure 3.39, there are two forward paths and three loops. There are no two nontouching loops. All the loops are touching one forward path and no loop is touching the second forward path as shown.

The forward paths and the gains associated with them are as follows:

Forward path R-x_1-x_2-x_3-x_4-x_5-x_6-C

$$M_1 = (1)(1)(G_1)(G_2)(G_3)(1)(1) = G_1G_2G_3 \qquad \Delta_1 = 1$$

Forward path R-x_1-x_6-C

$$M_2 = (1)(G_4)(1) = G_4 \qquad \Delta_2 = \Delta$$

The loops and the gains associated with them are as follows:

Loop x_3-x_4-x_5-x_3 $\quad L_1 = (G_2)(G_3)(-H_2) = -G_2G_3H_2$

Loop x_3-x_4-x_7-x_3 $\quad L_2 = (G_2)(H_1)(-1) = -G_2H_1$

Loop x_2-x_3-x_4-x_7-x_2 $\quad L_3 = (G_1)(G_2)(H_1)(1) = G_1G_2H_1$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3)$

$$\Delta = 1 - (-G_2G_3H_2 - G_2H_1 + G_1G_2H_1)$$

$$= 1 + G_2G_3H_2 + G_2H_1 - G_1G_2H_1$$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3}{1 + G_2H_1 + G_2G_3H_2 - G_1G_2H_1} + G_4$$

Example 3.5 Find the transfer function $\frac{C(s)}{R(s)}$ for the system shown in Figure 3.40 using the signal flow graph method.

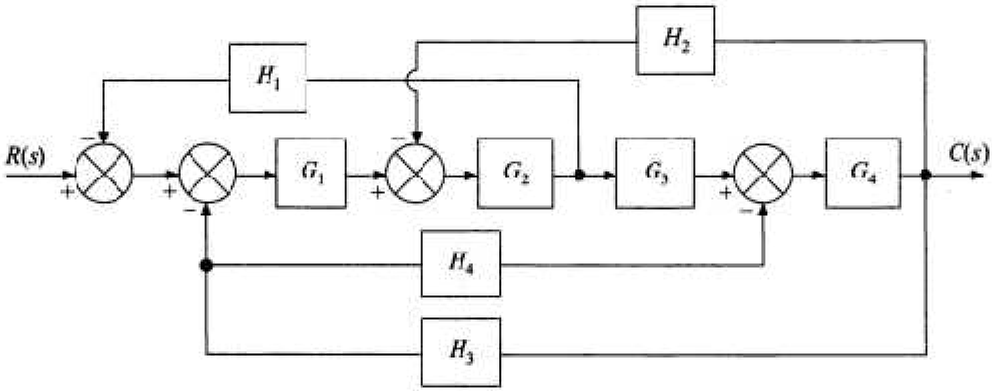


Figure 3.40 Example 3.5.

Solution: The signal flow graph corresponding to the given block diagram of Figure 3.40 is shown in Figure 3.41.

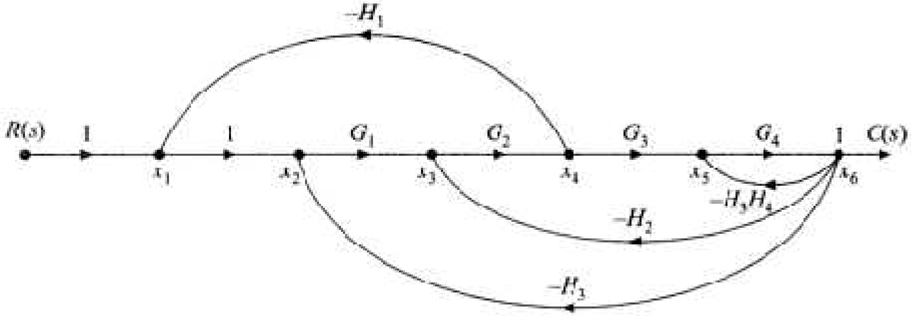


Figure 3.41

The signal flow graph shown in Figure 3.41 has only one forward path and four loops. There is one pair of two nontouching loops and all the four loops are touching the forward path.

The forward path and the gain associated with it are as follows:

Forward path $R(s)$-x_1-x_2-x_3-x_4-x_5-x_6-$C(s)$ $\quad M_1 = (1)(1)(G_1)(G_2)(G_3)(G_4)(1) = G_1G_2G_3G_4$

The loops and the gains associated with them are

Loop x_1-x_2-x_3-x_4-x_1 $\quad L_1 = (1)(G_1)(G_2)(-H_1) = -G_1G_2H_1$

Loop x_5-x_6-x_5 $\quad L_2 = (G_4)(-H_3H_4) = -G_4H_3H_4$

Loop x_3-x_4-x_5-x_6-x_3 $\quad L_3 = (G_2)(G_3)(G_4)(-H_2) = -G_2G_3G_4H_2$

Loop x_2-x_3-x_4-x_5-x_6-x_2 $\quad L_4 = (G_1)(G_2)(G_3)(G_4)(-H_3) = -G_1G_2G_3G_4H_3$

The pairs of two nontouching loops x_1-x_2-x_3-x_4-x_1 and x_5-x_6-x_5 and the products of gains associated with them are as follows:

$$L_{12} = (-G_1G_2H_1)(-G_4H_3H_4) = G_1G_2G_4H_1H_3H_4$$

$\Delta_1 = 1$ since all the loops are touching the only forward path.

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_{12})$$

The transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{M_1\Delta_1}{\Delta}$$

$$= \frac{G_1G_2G_3G_4}{1-(-G_1G_2H_1 - G_4H_3H_4 - G_2G_3G_4H_2 - G_1G_2G_3G_4H_3) + (G_1G_2G_4H_1H_3H_4)}$$

i.e. $$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1+G_1G_2H_1 + G_4H_3H_4 - G_2G_3G_4H_2 + G_1G_2G_3G_4H_3 + G_1G_2G_4H_1H_3H_4}$$

Example 3.6 For the system represented by the block diagram shown in Figure 3.42, evaluate the closed-loop transfer function when the input R is (a) at station A and (b) at station B.

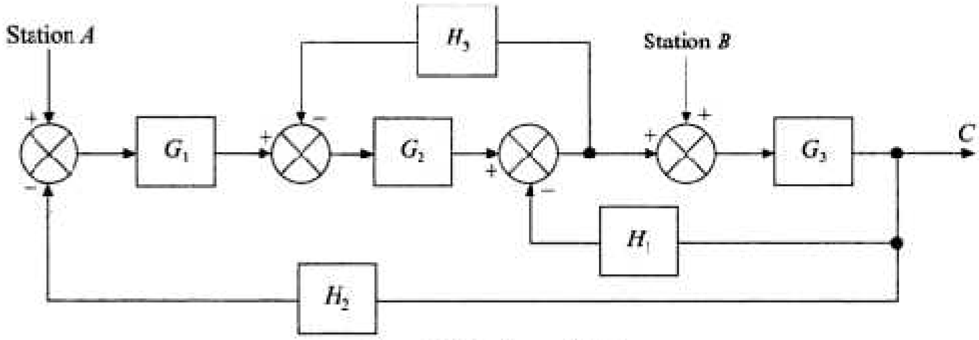


Figure 3.42 Example 3.6.

Solution:

(a) The input R is at station A and so the input at station B is made zero. Let the output be C_1. Since there is no input at station B, that summing point can be removed and the resultant block diagram will be as shown in Figure 3.43.

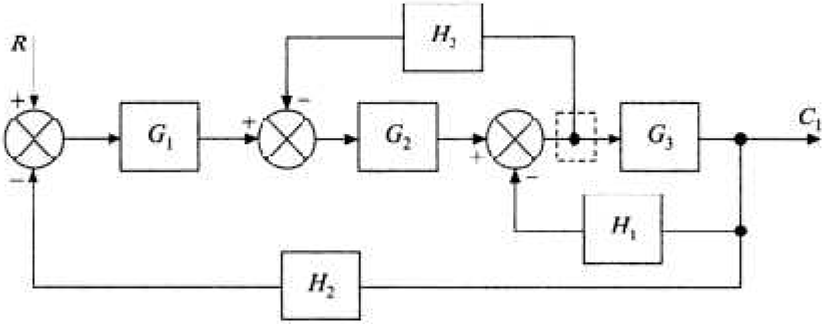


Figure 3.43

Step 1. Moving the take-off point in Figure 3.43 after block G_3, the block diagram will be as shown in Figure 3.44.

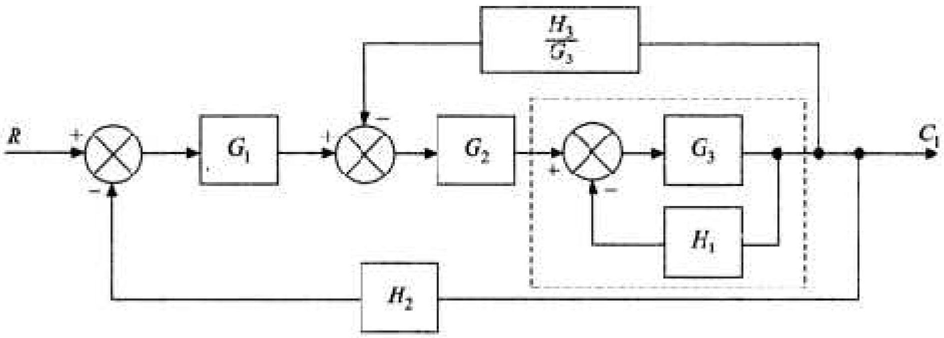


Figure 3.44

Step 2. Eliminating the inner loop in Figure 3.44 and combining the result with G_2 and rearranging, the block diagram will be as shown in Figure 3.45.

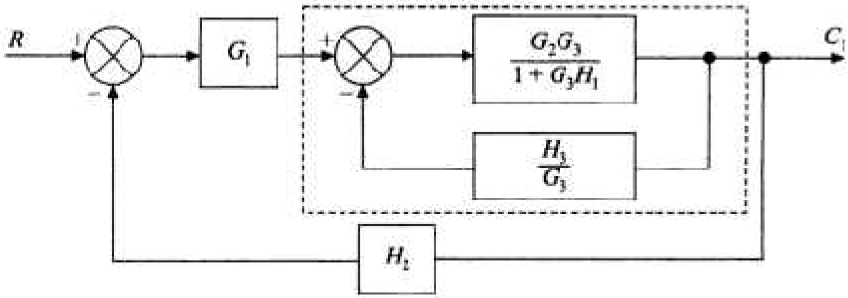


Figure 3.45

Step 3. Eliminating the inner loop in Figure 3.45, the block diagram is as shown in Figure 3.46.

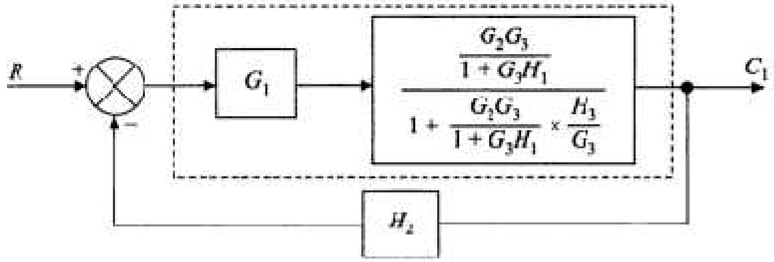


Figure 3.46

Step 4. Combining the blocks in cascade in Figure 3.46, the block diagram will be as shown in Figure 3.47.

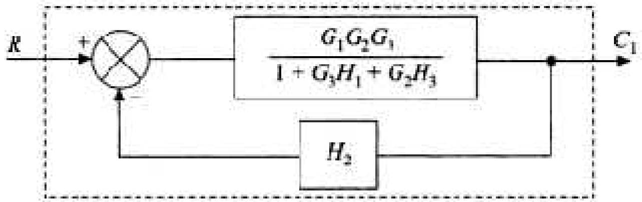


Figure 3.47

Step 5. Eliminating the only loop in Figure 3.47, the block diagram will be as shown in Figure 3.48.

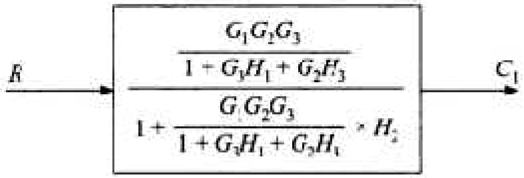


Figure 3.48

Step 6. After simplification, the closed-loop transfer function is

$$\frac{C_1}{R} = \frac{G_1G_2G_3}{1+G_3H_1+G_2H_3+G_1G_2G_3H_2}$$

(b) The input R is at station B. So the input at station A is made zero. Let the output be C_2. Since the input at station A is zero, the corresponding summing point can be removed and a negative sign can be attached to the feedback path gain H_2. The resulting block diagram is shown in Figure 3.49.

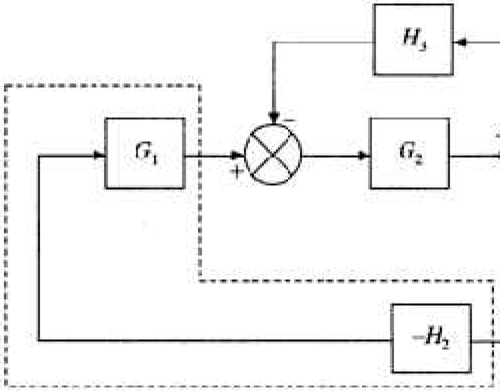


Figure 3.49

Step 1. Combining the blocks G_1 and $-H_2$ in cascade into a single block and rearranging the diagram in Figure 3.49, the resultant block diagram is as shown in Figure 3.50.

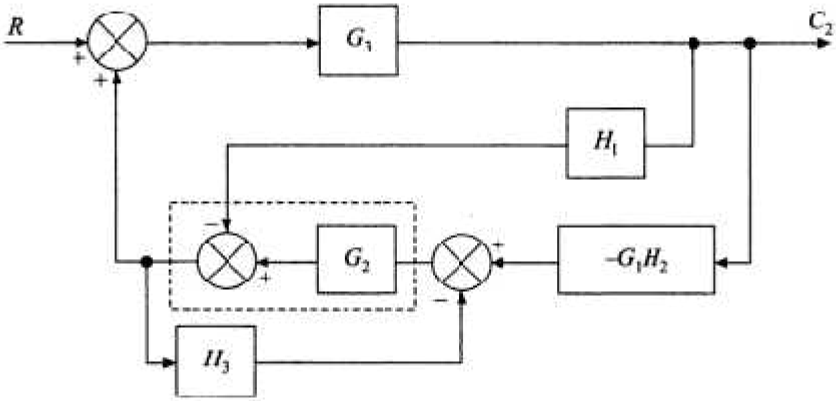


Figure 3.50

Step 2. Moving the summing point before the block G_2 in Figure 3.50, the block diagram will be as shown in Figure 3.51.

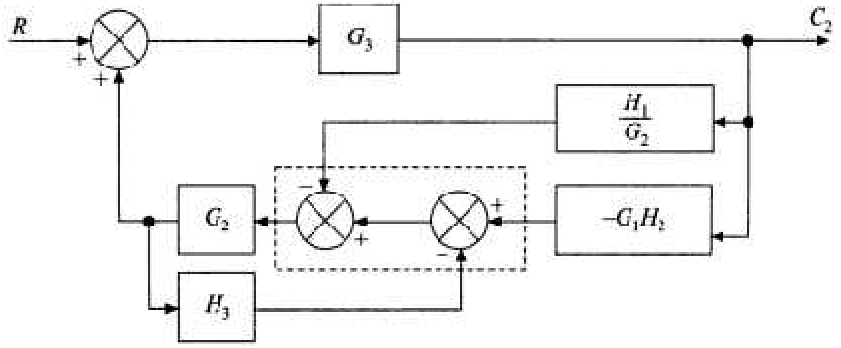


Figure 3.51

Step 3. Interchanging the summing points in Figure 3.51, the block diagram will be as shown in Figure 3.52.

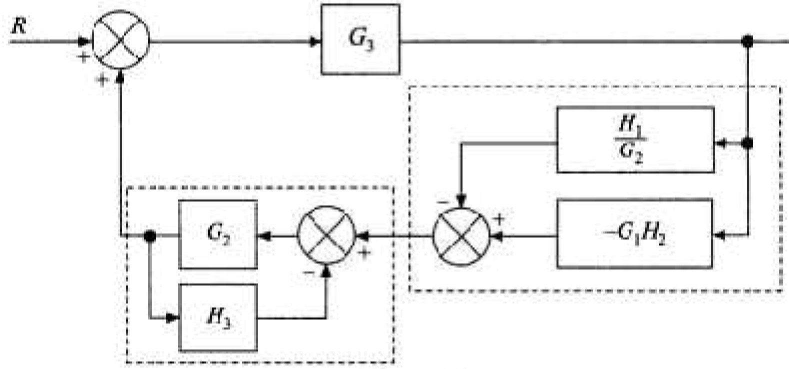


Figure 3.52

Step 4. Eliminating the loop and combining the parallel blocks in the feedback path in Figure 3.52, the block diagram will be as shown in Figure 3.53.

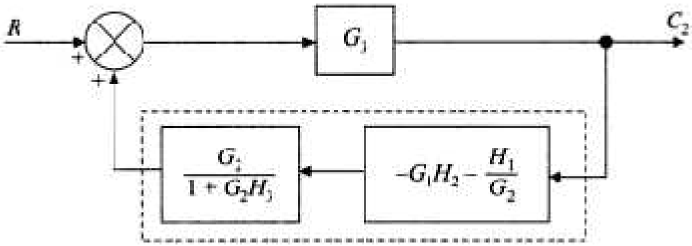


Figure 3.53

Step 5. Combining the two blocks in cascade in Figure 3.53, the block diagram will be as shown in Figure 3.54.

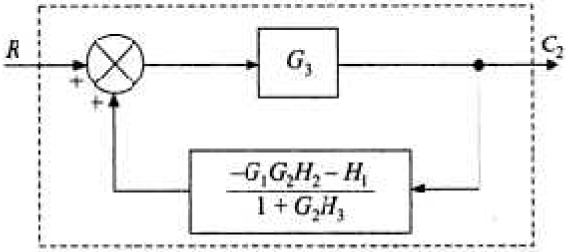


Figure 3.54

Step 6. Eliminating the loop in Figure 3.54, the transfer function will be as follows:

$$\frac{C_2}{R}=\frac{G_3}{1-\dfrac{G_3(-G_1G_2H_2-H_1)}{1+G_2H_3}}=\frac{G_3(1+G_2H_3)}{1+G_2H_3+G_3H_1+G_1G_2G_3H_2}$$

Example 3.7 For the system represented by the block diagram shown in Figure 3.55, determine (a) $\frac{C_1}{R_1}$ and (b) $\frac{C_2}{R_1}$.

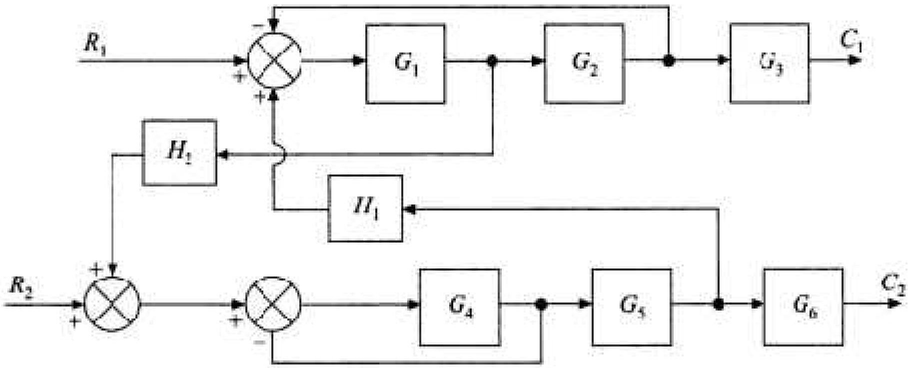


Figure 3.55 Example 3.7.

Solution: (a) *Determination of* $\frac{C_1}{R_1}$

To determine $\frac{C_1}{R_1}$, set $R_2 = 0$ and consider only one output C_1, and ignore C_2. Hence, the summing point which adds R_2 can be removed and there is no need to consider block G_6. With these modifications, the block diagram will be as shown in Figure 3.56.

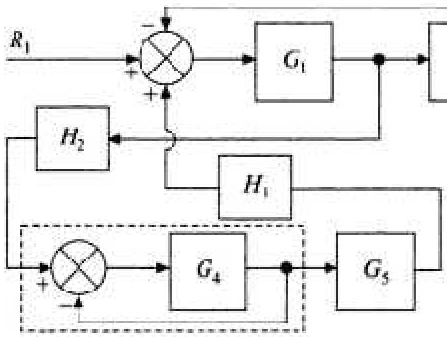


Figure 3.56

Step 1. Eliminating the feedback loop involving block G_4 in Figure 3.56, the block diagram will be as shown in Figure 3.57.

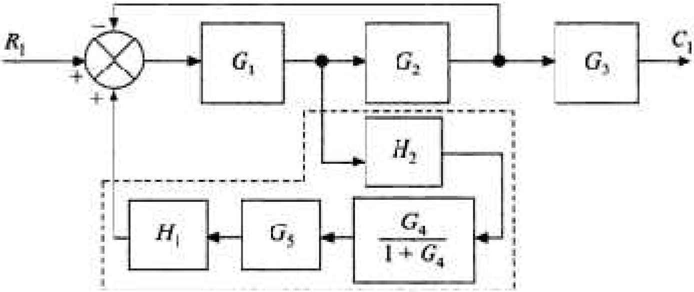


Figure 3.57

Step 2. Combining the blocks in cascade in the feedback path in Figure 3.57, the block diagram will be as shown in Figure 3.58.

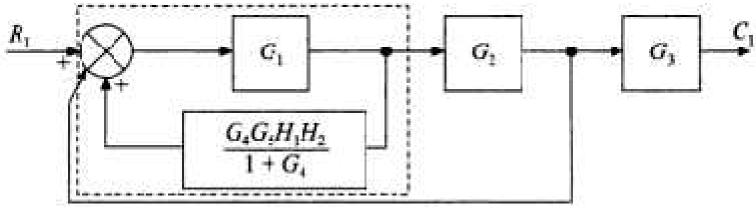


Figure 3.58

Step 3. Eliminating the inner loop in Figure 3.58, the block diagram will be as shown in Figure 3.59.

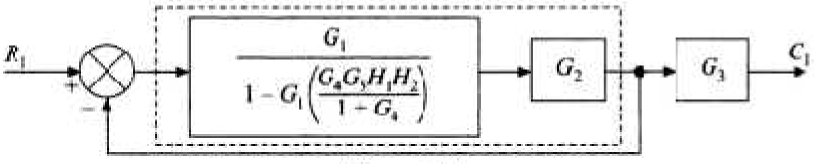


Figure 3.59

Step 4. Combining the blocks in cascade in Figure 3.59, the block diagram will be as shown in Figure 3.60.

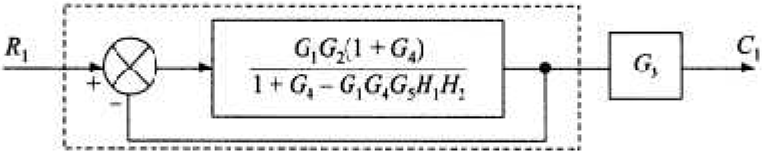


Figure 3.60

Step 5. Eliminating the only loop present in Figure 3.60, the block diagram will be as shown in Figure 3.61.

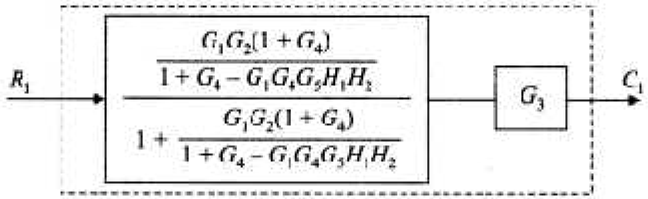


Figure 3.61

Step 6. Combining the blocks in cascade in Figure 3.61, the block diagram will be as shown in Figure 3.62.

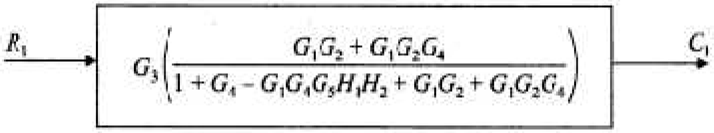


Figure 3.62

So the closed-loop transfer function is

$$\frac{C_1}{R_1}=\frac{G_1G_2G_3+G_1G_2G_3G_4}{1+G_4+G_1G_2+G_1G_2G_4-G_1G_4G_5H_1H_2}$$

(b) *Determination of* $\dfrac{C_2}{R_1}$

In this case, $R_2 = 0$ and C_1 need not be considered. Since $R_2 = 0$, we can remove the summing point which adds R_2 and since C_1 is not considered, block G_3 is open and can be ignored. The resultant block diagram is as shown in Figure 3.63.

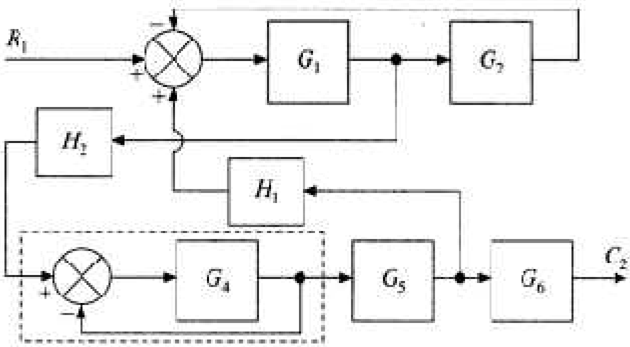


Figure 3.63

Step 1. Eliminating the unity feedback loop comprising G_4 in Figure 3.63. and redrawing Figure 3.63, the block diagram will be as shown in Figure 3.64.

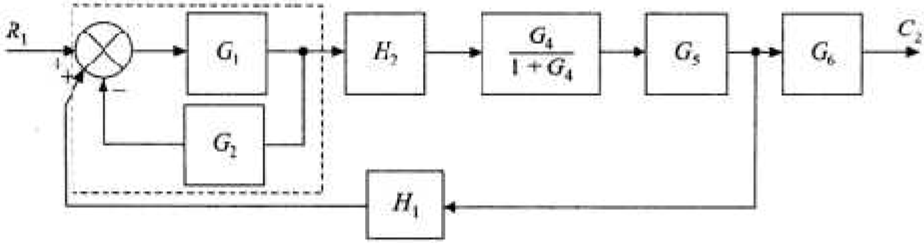


Figure 3.64

Step 2. Eliminating the inner loop comprising blocks G_1 and G_2 in Figure 3.64, the block diagram will be as shown in Figure 3.65.

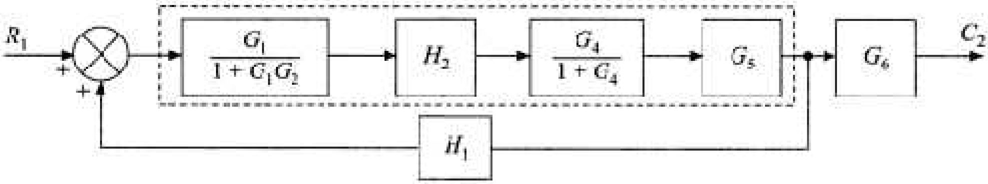


Figure 3.65

Step 3. Combining the blocks in cascade in Figure 3.65, the block diagram will be as shown in Figure 3.66.

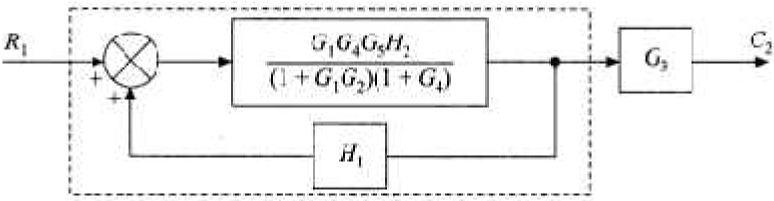


Figure 3.66

Step 4. Eliminating the only loop in Figure 3.66, the block diagram will be as shown in Figure 3.67.

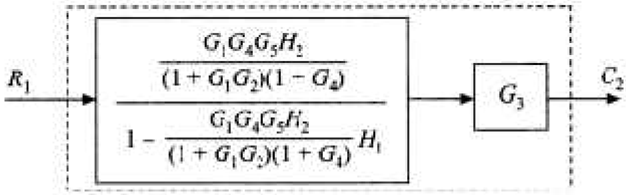


Figure 3.67

Step 5. Combining the blocks in cascade in Figure 3.67, the block diagram will be as shown in Figure 3.68.

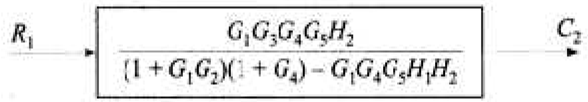


Figure 3.68

So the transfer function of the system will be given by

$$\frac{C_2(s)}{R_1(s)} = \frac{G_1G_3G_4G_5H_2}{1 + G_4 + G_1G_2 + G_1G_2G_4 - G_1G_4G_5H_1H_2}$$

Example 3.8 A system is described by the following set of linear algebraic equations:

$$x_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3$$
$$x_3 = a_{23}x_2 + a_{43}x_4$$
$$x_4 = a_{24}x_2 + a_{34}x_3 + a_{44}x_4$$
$$x_5 = a_{25}x_2 + a_{45}x_4$$

Draw the signal flow graph and obtain the transfer function of the system using Mason's gain formula.

Solution: The signal flow graph shown in Figure 3.69, drawn based on the given equations, has three forward paths and five loops. There are three pairs of two nontouching loops. All the loops are touching two forward paths and two loops are not touching the third forward path.

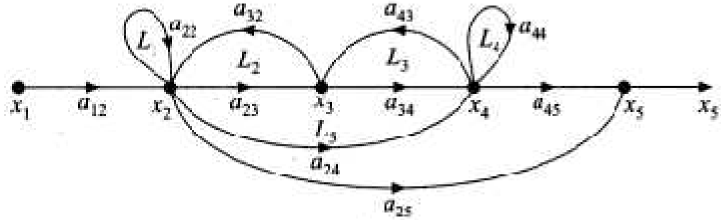


Figure 3.69 Example 3.8.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5

$$M_1 = (a_{12})(a_{23})(a_{34})(a_{45}) = a_{12}a_{23}a_{34}a_{45} \quad \Delta_1 = 1$$

Forward path x_1-x_2-x_4-x_5

$$M_2 = (a_{12})(a_{24})(a_{45}) = a_{12}a_{24}a_{45} \quad \Delta_2 = 1$$

Forward path x_1-x_2-x_5

$$M_3 = (a_{12})(a_{25}) = a_{12}a_{25} \quad \Delta_3 = 1 - (a_{34}a_{43} + a_{44})$$

The loops and the gains associated with them are as follows:

Loop x_2-x_2 $\quad L_1 = a_{22}$

Loop x_2-x_3-x_2 $\quad L_2 = (a_{23})(a_{32}) = a_{23}a_{32}$

Loop x_3-x_4-x_3 $\quad L_3 = (a_{34})(a_{43}) = a_{34}a_{43}$

Loop x_4-x_4 $\quad L_4 = a_{44}$

Loop x_2-x_4-x_3-x_2 $\quad L_5 = (a_{24})(a_{43})(a_{32}) = a_{24}a_{43}a_{32}$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops L_1 and L_3 $\quad L_{13} = (a_{22})(a_{34}a_{43}) = a_{22}a_{34}a_{43}$

Loops L_1 and L_4 $\quad L_{14} = (a_{22})(a_{44}) = a_{22}a_{44}$

Loops L_2 and L_4 $\quad L_{24} = (a_{23}a_{32})(a_{44}) = a_{23}a_{32}a_{44}$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{13} + L_{14} + L_{24})$$

$$\therefore \quad \Delta = 1 - (a_{22} + a_{23}a_{32} + a_{34}a_{43} + a_{44} + a_{24}a_{43}a_{32}) + (a_{22}a_{34}a_{43} + a_{22}a_{44} + a_{23}a_{32}a_{44})$$

Applying Mason's gain formula, the transfer function is

$$T = \frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

$$= \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{24}a_{45} + a_{12}a_{25}(1 - a_{34}a_{43} - a_{44})}{1 - a_{22} - a_{23}a_{32} - a_{34}a_{43} - a_{44} - a_{24}a_{43}a_{32} + a_{22}a_{34}a_{43} + a_{22}a_{44} + a_{23}a_{32}a_{44}}$$

Example 3.9 For the system represented by the following equations, find the transfer function $X(s)/U(s)$ by the signal flow graph technique.

$$x = x_1 + \alpha_0 u$$

$$\frac{dx_1}{dt} = -\alpha_1 x_1 + x_2 + \alpha_2 u$$

$$\frac{dx_2}{dt} = -\alpha_2 x_1 + \alpha_1 u$$

Solution: The given equations are a set of first-order differential equations. For the signal flow graph, the describing equations must be linear algebraic equations. The differential equations may be converted into algebraic equations by using the Laplace transform.

$$X(s) = X_1(s) + \alpha_0 U(s)$$
$$sX_1(s) = -\alpha_1 X_1(s) + X_2(s) + \alpha_2 U(s)$$
$$sX_2(s) = -\alpha_2 X_1(s) + \alpha_1 U(s)$$

i.e.
$$X(s) = X_1(s) + \alpha_0 U(s)$$
$$X_1(s) = \frac{-\alpha_1}{s} X_1(s) + \frac{1}{s} X_2(s) + \frac{\alpha_2}{s} U(s)$$
$$X_2(s) = \frac{-\alpha_2}{s} X_1(s) + \frac{\alpha_1}{s} U(s)$$

Based on these equations, the signal flow graph is drawn as shown in Figure 3.70.

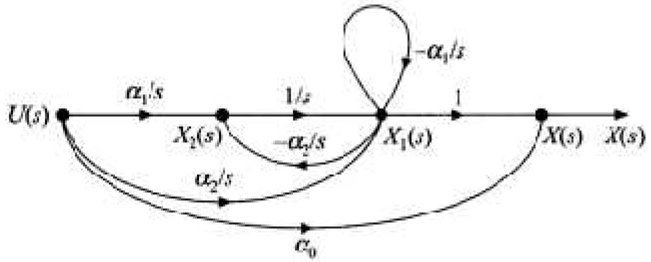


Figure 3.70 Example 3.9.

In the signal flow graph of Figure 3.70, there are three forward paths and two loops. There are no pairs of nontouching loops. Both the loops are touching two forward paths and both the loops are not touching the third forward path.

The forward paths and the gains associated with them are as follows:

Forward path $U(s)$-$X_2(s)$-$X_1(s)$-$X(s)$ $\quad M_1 = \frac{\alpha_1}{s} \cdot \frac{1}{s} \cdot 1 = \frac{\alpha_1}{s^2}$

Forward path $U(s)$-$X_1(s)$-$X(s)$ $\quad M_2 = \frac{\alpha_2}{s} \cdot 1 = \frac{\alpha_2}{s}$

Forward path $U(s)$-$X(s)$ $\quad M_3 = \alpha_0$

The loops and the gains associated with them are as follows:

Loop $X_2(s)$-$X_1(s)$-$X_2(s)$ $\quad L_1 = \left(\frac{-\alpha_2}{s}\right)\left(\frac{1}{s}\right) = \frac{-\alpha_2}{s^2}$

Loop $X_1(s)$-$X_1(s)$ $\quad L_2 = \frac{-\alpha_1}{s}$

Both the loops are touching the first and second forward paths and no loop is touching the third forward path

$$\therefore \quad \Delta_1 = 1, \Delta_2 = 1, \Delta_3 = 1 - \left(-\frac{\alpha_2}{s^2} - \frac{\alpha_1}{s}\right) = 1 + \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2} = \Delta$$

The determinant of the signal flow graph is D = 1 – (L_1 + L_2)

$$\therefore \qquad \Delta = 1 - \left(-\frac{\alpha_2}{s^2} - \frac{\alpha_1}{s}\right) = 1 + \frac{\alpha_1}{s} + \frac{\alpha_2}{s^2}$$

Using Mason's gain formula, the transfer function is

$$\frac{X(s)}{U(s)} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta} = \frac{\dfrac{\alpha_1}{s^2} + \dfrac{\alpha_2}{s}}{1 + \dfrac{\alpha_1}{s} + \dfrac{\alpha_2}{s^2}} + \alpha_0 = \alpha_0 + \frac{\alpha_1 + \alpha_2 s}{s^2 + \alpha_1 s + \alpha_2}$$

$$\frac{X(s)}{U(s)} = \frac{\alpha_0(s^2 + \alpha_1 s + \alpha_2) + \alpha_1 + \alpha_2 s}{s^2 + \alpha_1 s + \alpha_2}$$

Example 3.10 Apply the gain formula to the signal flow graph shown in Figure 3.71, to find the following transfer functions:

$$\frac{x_5}{x_1}, \frac{x_4}{x_1}, \frac{x_2}{x_1}, \frac{x_5}{x_2}, \frac{x_4}{x_2}$$

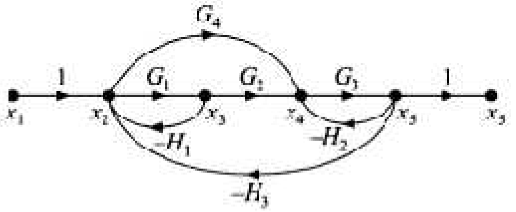


Figure 3.71 Exampe 3.10.

Solution: The signal flow graph shown in Figure 3.71 has two forward paths, four loops, and one pair of two nontouching loops. All the loops are touching both the forward paths.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5 $\quad M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3 \quad \Delta_1 = 1$

Forward path x_1-x_2-x_4-x_5 $\quad M_2 = (1)(G_4)(G_3)(1) = G_4G_3 \quad \Delta_2 = 1$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $\quad L_1 = (G_1)(-H_1) = -G_1H_1$

Loop x_4-x_5-x_4 $\quad L_2 = (G_3)(-H_2) = -G_3H_2$

Loop x_2-x_3-x_4-x_5-x_2 $\quad L_3 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$

Loop x_2-x_4-x_5-x_2 $\quad L_4 = (G_4)(G_3)(-H_3) = -G_4G_3H_3$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2 and x_4-x_5-x_4 $\quad L_{12} = (-G_1H_1)(-G_3H_2) = G_1G_3H_1H_2$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4) + (L_{12})$

$\therefore \quad \Delta = 1 - (-G_1H_1 - G_3H_2 - G_1G_2G_3H_3 - G_3G_4H_3) + G_1G_3H_1H_2$

$= 1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_3G_4H_3 + G_1G_3H_1H_2$

The transfer function is

$$\frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_3G_4}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_3G_4H_3 + G_1G_3H_1H_2}$$

To find the transfer function $\frac{x_4}{x_1}$

There are two forward paths between x_1 and x_4 and there are no loops which are not touching these forward paths

Forward path x_1-x_2-x_3-x_4 $\quad M_1 = (1)(G_1)(G_2) = G_1G_2 \quad \Delta_1 = 1$

Forward path x_1-x_2-x_4 $\quad M_2 = (1)(G_4) = G_4 \quad \Delta_2 = 1$

$$\therefore \quad \frac{x_4}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2 + G_4}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_3G_4H_3 + G_1G_3H_1H_2}$$

To find the transfer function $\frac{x_2}{x_1}$

There is only one forward path between x_1 and x_2. The loop L_2 is not touching this forward path

$M_1 = 1,\ \Delta_1 = 1 - (-G_3H_2) = 1 + G_3H_2$

$$\therefore \quad \frac{x_2}{x_1} = \frac{M_1\Delta_1}{\Delta} = \frac{1\cdot(1 + G_3H_2)}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_3G_4H_3 + G_1G_3H_1H_2}$$

To find the transfer function $\frac{x_5}{x_2}$

Find $\frac{x_5}{x_1}$ and $\frac{x_2}{x_1}$ and divide one by the other, i.e.

$$\frac{x_5}{x_2} = \frac{\frac{x_5}{x_1}}{\frac{x_2}{x_1}} = \frac{\frac{G_1G_2G_3 + G_3G_4}{\Delta}}{\frac{1 + G_3H_2}{\Delta}} = \frac{G_1G_2G_3 + G_3G_4}{1 + G_3H_2}$$

To find the transfer function $\frac{x_4}{x_2}$

Find $\frac{x_4}{x_1}$ and $\frac{x_2}{x_1}$ and divide one by the other, i.e.

$$\frac{x_4}{x_2} = \frac{\frac{x_4}{x_1}}{\frac{x_2}{x_1}} = \frac{\frac{G_1G_2 + G_4}{\Delta}}{\frac{1 + G_3H_2}{\Delta}} = \frac{G_1G_2 + G_4}{1 + G_3H_2}$$

Example 3.11 Apply the gain formula to the signal flow graph shown in Figure 3.72.

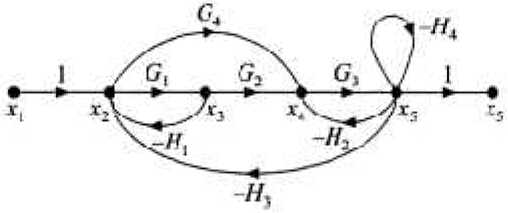


Figure 3.72 Example 3.11.

Solution: In the signal flow graph shown in Figure 3.72, there are two forward paths, five loops and two pairs of two nontouching loops. All the loops are touching both the forward paths.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5 $\quad M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3 \quad \Delta_1 = 1$

Forward path x_1-x_2-x_4-x_5 $\quad M_2 = (1)(G_4)(G_3)(1) = G_4G_3 \quad \Delta_2 = 1$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $\quad L_1 = (G_1)(-H_1) = -G_1H_1$

Loop x_4-x_5-x_4 $\quad L_2 = (G_3)(-H_2) = -G_3H_2$

Loop x_2-x_3-x_4-x_5-x_2 $\quad L_3 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$

Loop x_2-x_4-x_5-x_2 $\quad L_4 = (G_4)(G_3)(-H_3) = -G_4G_3H_3$

Loop x_5-x_5 $\quad L_5 = -H_4$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2 and x_4-x_5-x_4 $\quad L_{12} = (-G_1H_1)(-G_3H_2) = G_1G_3H_1H_2$

Loops x_2-x_3-x_2 and x_5-x_5 $\quad L_{15} = (-G_1\ H_1)(-H_4) = G_1H_1H_4$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{12} + L_{15})$

$$\Delta = 1 - (-G_1H_1 - G_3H_2 - G_1G_2G_3H_3 - G_4G_3H_3 - H_4) + (G_1G_3H_1H_2 + G_1H_1H_4)$$

$$= 1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_4G_3H_3 + H_4 + G_1G_3H_1H_2 + G_1H_1H_4$$

Applying Mason's gain formula, the transfer function is

$$\frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3 + G_4G_3}{1 + G_1H_1 + G_3H_2 + G_1G_2G_3H_3 + G_4G_3H_3 + H_4 + G_1G_3H_1H_2 + G_1H_1H_4}$$

Example 3.12 Find the transfer function $\frac{x_5}{x_1}$, for the system whose signal flow graph is shown in Figure 3.73, by applying the gain formula.

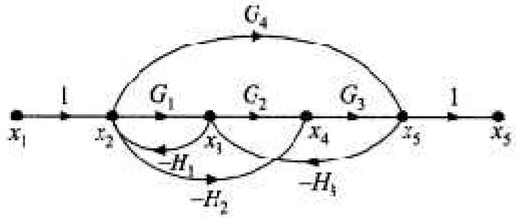


Figure 3.73 Example 3.12.

Solution: The signal flow graph shown in Figure 3.73 has two forward paths and five loops. There are no pairs of two loops which are not touching each other, and all the loops are touching both the forward paths.

The forward paths and the gains associated with them are as follows:

Forward path	x_1-x_2-x_3-x_4-x_5	$M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3$	$\Delta_1 = 1$
Forward path	x_1-x_2-x_5	$M_2 = (1)(G_4)(1) = G_4$	$\Delta_2 = 1$

The loops and the gains associated with them are as follows:

Loop	x_2-x_3-x_2	$L_1 = (G_1)(-H_1) = -G_1H_1$
Loop	x_2-x_3-x_4-x_2	$L_2 = (G_1)(G_2)(-H_2) = -G_1G_2H_2$
Loop	x_3-x_4-x_5-x_3	$L_3 = (G_2)(G_3)(-H_3) = -G_2G_3H_3$
Loop	x_2-x_5-x_3-x_2	$L_4 = (G_4)(-H_3)(-H_1) = G_4H_3H_1$
Loop	x_2-x_5-x_3-x_4-x_2	$L_5 = (G_4)(-H_3)(G_2)(-H_2) = G_4G_2H_3H_2$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$

$$\Delta = 1 - (-G_1H_1 - G_1G_2H_2 - G_2G_3H_3 + G_4G_3H_1 + G_4G_2H_3H_2)$$
$$= 1 + G_1H_1 + G_1G_2H_2 + G_2G_3H_3 - G_4G_3H_1 - G_4G_2H_3H_2$$

Applying Mason's gain formula, the transfer function is

$$\frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4}{1 + G_1H_1 + G_1G_2H_2 + G_2G_3H_3 - G_4H_3H_1 - G_4G_2H_3H_2}$$

Example 3.13 Apply the gain formula to the signal flow graph shown in Figure 3.74, to find the transfer function $\frac{x_5}{x_1}$.

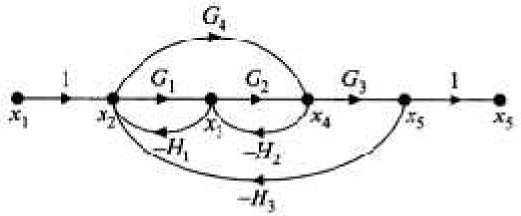


Figure 3.74 Example 3.13.

Solution: The signal flow graph shown in Figure 3.74 has two forward paths, and five loops. There are no pairs of two nontouching loops and all the loops are touching both the forward paths.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5 $\quad M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3 \quad \Delta_1 = 1$

Forward path x_1-x_2-x_4-x_5 $\quad M_2 = (1)(G_4)(G_3)(1) = G_4G_3 \quad \Delta_2 = 1$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $\quad L_1 = (G_1)(-H_1) = -G_1H_1$

Loop x_3-x_4-x_3 $\quad L_2 = (G_2)(-H_2) = -G_2H_2$

Loop x_2-x_4-x_3-x_2 $\quad L_3 = (G_4)(-H_2)(-H_1) = G_4H_2H_1$

Loop x_2-x_4-x_5-x_2 $\quad L_4 = (G_4)(G_3)(-H_3) = -G_4G_3H_3$

Loop x_2-x_3-x_4-x_5-x_2 $\quad L_5 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$

$$\Delta = 1 - (-G_1H_1 - G_2H_2 + G_4H_2H_1 - G_4G_3H_2 - G_1G_2G_3H_3)$$

Applying Mason's gain formula, the transfer function is

$$\frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3 + G_4G_3}{1 + G_1H_1 + G_2H_2 + G_4G_3H_3 + G_1G_2G_3H_3 - G_4H_2H_1}$$

Example 3.14 Apply the gain formula to the signal flow graph shown in Figure 3.75, to find the transfer function $\dfrac{x_5}{x_1}$.

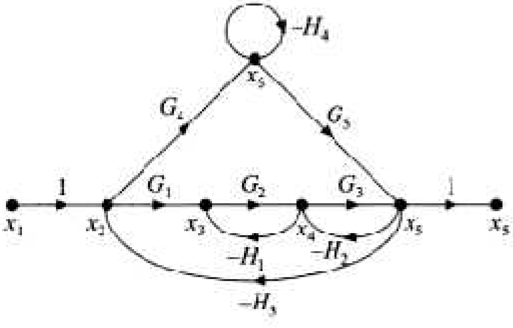


Figure 3.75 Example 3.14.

Solution: The signal flow graph shown in Figure 3.75 has two forward paths and five loops. There are four pairs of two nontouching loops and each forward path has got one nontouching loop.

The forward paths and gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5 $\quad M_1 = (1)(G_1)(G_2)(G_3)(1) = G_1G_2G_3$

Forward path x_1-x_2-x_6-x_5 $\quad M_2 = (1)(G_4)(G_5)(1) = G_4G_5$

The loops and the gains associated with them are

Loop x_3-x_4-x_3	$L_1 = (G_2)(-H_1) = -G_2H_1$
Loop x_4-x_5-x_4	$L_2 = (G_3)(-H_2) = -G_3H_2$
Loop x_2-x_3-x_4-x_5-x_2	$L_3 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$
Loop x_6-x_6	$L_4 = -H_4$
Loop x_2-x_6-x_5-x_2	$L_5 = (G_4)(G_5)(-H_3) = -G_4G_5H_3$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_3-x_4-x_3 and x_6-x_6	$L_{14} = (-G_2H_1)(-H_4) = G_2H_1H_4$
Loops x_3-x_4-x_3 and x_2-x_6-x_5-x_2	$L_{15} = (-G_2H_1)(-G_4G_5H_3) = G_2H_1G_4G_5H_3$
Loops x_4-x_5-x_4 and x_6-x_6	$L_{24} = (-G_3H_2)(-H_4) = G_3H_2H_4$
Loops x_2-x_3-x_4-x_5-x_2 and x_6-x_6	$L_{34} = (-G_1G_2G_3H_3)(-H_4) = G_1G_2G_3H_3H_4$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{14} + L_{15} + L_{24} + L_{34})$$

$$\Delta = 1 - (-G_2H_1 - G_3H_2 - G_1G_2G_3H_3 - H_4 - G_4G_5H_3) + (G_2H_1H_4 + G_2H_1G_4G_5H_3 + G_3H_2H_4 + G_1G_2G_3H_3H_4)$$

$$= 1 + G_2H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_4G_5H_3 + G_2H_1H_4 + G_2H_1G_4G_5H_3 + G_3H_2H_4 + G_1G_2G_3H_3H_4$$

L_4 is not touching the first forward path and L_1 is not touching the second forward path

$\therefore \qquad \Delta_1 = 1 - (-H_4) = 1 + H_4$

and $\qquad \Delta_2 = 1 - (-G_2H_1) = 1 + G_2H_1$

Applying Mason's gain formula, the transfer function is

$$\frac{x_5}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta}$$

$$= \frac{G_1G_2G_3(1 + H_4) + G_4G_5(1 + G_2H_1)}{1 + G_2H_1 + G_3H_2 + G_1G_2G_3H_3 + H_4 + G_4G_5H_3 + G_2H_1H_4 + G_2H_1G_4G_5H_3 + G_3H_2H_4 + G_1G_2G_3H_3H_4}$$

Example 3.15 Find the transfer function $\frac{x_7}{x_1}$ of the signal flow graph shown in Figure 3.76.

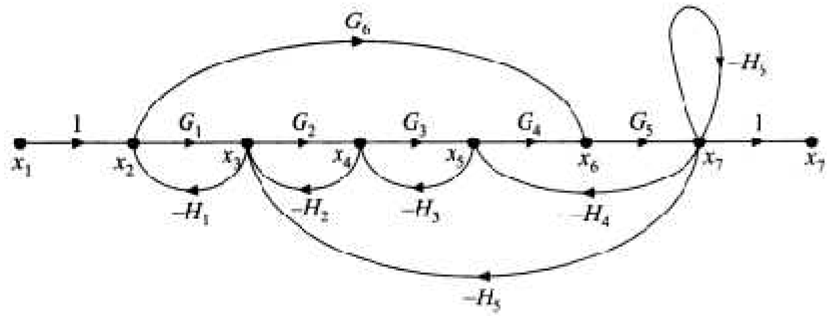


Figure 3.76 Example 3.15.

Solution: The signal flow graph shown in Figure 3.76 has two forward paths and eight loops. There are seven pairs of two nontouching loops. There is one combination of three nontouching loops. All loops are touching one forward path and two loops are not touching the second forward path.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5-x_6-x_7 $\quad M_1 = (1)(G_1)(G_2)(G_3)(G_4)(G_5)(1) = G_1G_2G_3G_4G_5$

Forward path x_1-x_2-x_6-x_7 $\quad M_2 = (1)(G_6)(G_5)(1) = G_6G_5$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $\quad L_1 = (G_1)(-H_1) = -G_1H_1$

Loop x_3-x_4-x_3 $\quad L_2 = (G_2)(-H_2) = -G_2H_2$

Loop x_4-x_5-x_4 $\quad L_3 = (G_3)(-H_3) = -G_3H_3$

Loop x_5-x_6-x_7-x_5 $\quad L_4 = (G_4)(G_5)(-H_4) = -G_4G_5H_4$

Loop x_7-x_7 $\quad L_5 = -H_6$

Loop x_3-x_4-x_5-x_6-x_7-x_3 $\quad L_6 = (G_2)(G_3)(G_4)(G_5)(-H_5) = -G_2G_3G_4G_5H_5$

Loop x_2-x_6-x_7-x_5-x_4-x_3-x_2 $\quad L_7 = (G_6)(G_5)(-H_4)(-H_3)(-H_2)(-H_1) = G_6G_5H_4H_3H_2H_1$

Loop x_2-x_6-x_7-x_3-x_2 $\quad L_8 = (G_6)(G_5)(-H_5)(-H_1) = G_6G_5H_5H_1$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2 and x_4-x_5-x_4 $\quad L_{13} = (-G_1H_1)(-G_3H_3) = G_1H_1G_3H_3$

Loops x_2-x_3-x_2 and x_5-x_6-x_7-x_5 $\quad L_{14} = (-G_1H_1)(-G_4G_5H_4) = G_1H_1G_4G_5H_4$

Loops x_2-x_3-x_2 and x_7-x_7 $\quad L_{15} = (-G_1H_1)(-H_6) = G_1H_1H_6$

Loops x_3-x_4-x_3 and x_5-x_6-x_7-x_5 $\quad L_{24} = (-G_2H_2)(-G_4G_5H_4) = G_2H_2G_4G_5H_4$

Loops x_3-x_4-x_3 and x_7-x_7 $\quad L_{25} = (-G_2H_2)(-H_6) = G_2H_2H_6$

Loops x_4-x_5-x_4 and x_7-x_7 $\quad L_{35} = (-G_3H_3)(-H_6) = G_3H_3H_6$

Loops x_4-x_5-x_4 and x_2-x_6-x_7-x_3-x_2 $\quad L_{38} = (-G_3H_3)(G_6G_5H_5H_1) = -G_3H_3G_6G_5H_5H_1$

The combinations of three nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2, x_4-x_5-x_4 and x_7-x_7 $\quad L_{135} = (-G_1H_1)(-G_3H_3)(-H_6) = -G_1H_1G_3H_3H_6$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_8) + (L_{13} + L_{14} + L_{15} + L_{24} + L_{25} + L_{35} + L_{38}) - (L_{135})$$

$$\therefore \quad \Delta = 1 - (-G_1H_1 - G_2H_2 - G_3H_3 - G_4G_5H_4 - H_6 - G_2G_3G_4G_5H_5 + G_6G_5H_4H_3H_2H_1 + G_6G_5H_5H_1) + (G_1H_1G_3H_3 + G_1H_1G_4G_5H_4 + G_1H_1H_6 + G_2H_2H_6 + G_3H_3H_6 - G_3H_3G_6G_5H_5H_1 + G_2H_2G_4G_5H_4) - (-G_1H_1G_3H_3H_6)$$

All loops are touching the first forward path and L_2 and L_6 are not touching the second forward path

$$\therefore \quad \Delta_1 = 1 \text{ and } \Delta_2 = 1 - (-G_2H_2 - G_3H_3)$$

Applying the Mason's gain formula, the transfer function is

$$\frac{x_7}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_6G_5(1+G_2H_2+G_3H_3)}{\Delta}$$

Example 3.16 Find the transfer function $\frac{x_7}{x_1}$ of the signal flow graph shown in Figure 3.77.

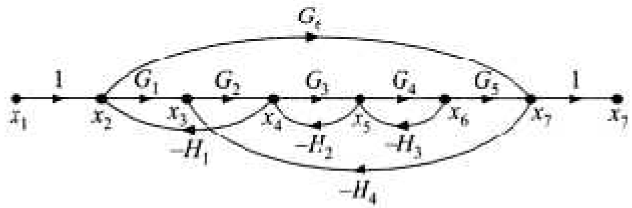


Figure 3.77 Example 3.16.

Solution: The signal flow graph shown in Figure 3.77 has two forward paths and five loops. There are two pairs of loops which are not touching each other. All the loops are touching one forward path and two loops are not touching the second forward path.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5-x_6-x_7 $\quad M_1 = (1)(G_1)(G_2)(G_3)(G_4)(G_5)(1) = G_1G_2G_3G_4G_5$

Forward path x_1-x_2-x_7 $\quad M_2 = (1)(G_6)(1) = G_6$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_4-x_2 $\quad L_1 = (G_1)(G_2)(-H_1) = -G_1G_2H_1$

Loop x_4-x_5-x_4 $\quad L_2 = (G_3)(-H_2) = -G_3H_2$

Loop x_5-x_6-x_5 $\quad L_3 = (G_4)(-H_3) = -G_4H_3$

Loop x_3-x_4-x_5-x_6-x_7-x_3 $\quad L_4 = (G_2)(G_3)(G_4)(G_5)(-H_4) = -G_2G_3G_4G_5H_4$

Loop x_2-x_7-x_3-x_4-x_2 $\quad L_5 = (G_6)(-H_4)(G_2)(-H_1) = G_2G_6H_1H_4$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_4-x_2 and x_5-x_6-x_5 $\quad L_{13} = (-G_1G_2H_1)(-G_4H_3) = G_1G_2H_1G_4H_3$

Loops x_5-x_6-x_5 and x_2-x_7-x_3-x_4-x_2 $\quad L_{35} = (-G_4H_3)(G_6G_2H_4H_1) = -G_4H_3G_6G_2H_4H_1$

The determinant of the signal flow graph is $\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{13} + L_{35})$

$$\Delta = 1 - (-G_1G_2H_1 - G_3H_2 - G_4H_3 - G_2G_3G_4G_5H_4 + G_2G_6H_1H_4)$$
$$+ (G_1G_2H_1G_4H_3 - G_4G_3G_6G_2H_4H_1)$$

All loops are touching the first forward path and L_2 and L_3 are not touching the second forward path

$\therefore \quad \Delta_1 = 1$ and $\Delta_2 = 1 - (-G_3H_2 - G_4H_3) = 1 + G_3H_2 + G_4H_3$

Applying Mason's gain formula, the transfer function is

$$\frac{x_7}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_6(1+G_3H_2+G_4H_3)}{\Delta}$$

Example 3.17 Find the transfer function $\frac{C}{R}$ of the signal flow graph shown in Figure 3.78.

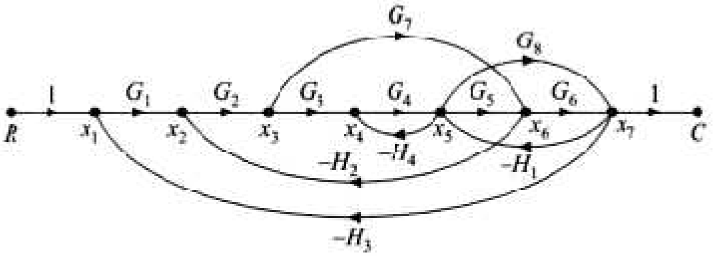


Figure 3.78 Example 3.17.

Solution: The signal flow graph shown in Figure 3.78 has three forward paths and eight loops. There are three pairs of two nontouching loops. All the loops are touching two forward paths and one loop is not touching the third forward path.

The forward paths and gains associated with them are as follows:

Forward path R-x_1-x_2-x_3-x_4-x_5-x_6-x_7-C $M_1 = (1)(G_1)(G_2)(G_3)(G_4)(G_5)(G_6)(1)$
$= G_1G_2G_3G_4G_5G_6$

Forward path R-x_1-x_2-x_3-x_6-x_7-C $M_2 = (1)(G_1)(G_2)(G_7)(G_6)(1) = G_1G_2G_7G_6$

Forward path R-x_1-x_2-x_3-x_4-x_5-x_7-C $M_3 = (1)(G_1)(G_2)(G_3)(G_4)(G_8)(1)$
$= G_1G_2G_3G_4G_8$

The loops and the gains associated with them are as follows:

Loop x_4-x_5-x_4 $L_1 = (G_4)(-H_4) = -G_4H_4$

Loop x_2-x_3-x_4-x_5-x_6-x_2 $L_2 = (G_2)(G_3)(G_4)(G_5)(-H_2)$
$= -G_2G_3G_4G_5H_2$

Loop x_2-x_3-x_6-x_2 $L_3 = (G_2)(G_7)(-H_2) = -G_2G_7H_2$

Loop x_5-x_6-x_7-x_5 $L_4 = (G_5)(G_6)(-H_1) = -G_5G_6H_1$

Loop x_5-x_7-x_5 $L_5 = (G_8)(-H_1) = -G_8H_1$

Loop x_1-x_2-x_3-x_4-x_5-x_6-x_7-x_1 $L_6 = (G_1)(G_2)(G_3)(G_4)(G_5)(G_6)(-H_3)$
$= -G_1G_2G_3G_4G_5G_6H_3$

Loop x_1-x_2-x_3-x_6-x_7-x_1 $L_7 = (G_1)(G_2)(G_7)(G_6)(-H_3)$
$= -G_1G_2G_7G_6H_3$

Loop x_1-x_2-x_3-x_4-x_5-x_7-x_1 $L_8 = (G_1)(G_2)(G_3)(G_4)(G_8)(-H_3)$
$= -G_1G_2G_3G_4G_8H_3$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_4-x_5-x_4 and x_2-x_3-x_6-x_2 $L_{13} = (-G_4H_4)(-G_2G_7H_2) = G_4H_4G_2G_7H_2$

Loops x_4-x_5-x_4 and x_1-x_2-x_3-x_6-x_7-x_1 $\quad L_{17} = (-G_4H_4)(-G_1G_2G_7G_6H_3)$

$= G_4H_4G_1G_2G_7G_6H_3$

Loops x_2-x_3-x_6-x_2 and x_5-x_7-x_5 $\quad L_{35} = (-G_2G_7H_2)(-G_8H_1) = G_2G_7H_2G_8H_1$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + L_{18}) + (L_{13} + L_{17} + L_{35})$$

$$\begin{aligned}\Delta = 1 - (&-G_4H_4 - G_2G_3G_4G_5H_2 - G_2G_7H_2 - G_5G_6H_1 - G_8H_1 \\ &- G_1G_2G_3G_4G_5G_6H_3 - G_1G_2G_7G_6H_3 - G_1G_2G_3G_4G_8H_3) \\ &+ (G_4H_4G_2G_7H_2 + G_4H_4G_1G_2G_7G_6H_3 + G_2G_7H_2G_8H_1)\end{aligned}$$

All loops are touching the first and third forward paths and L_1 is not touching the second forward path

$\therefore \quad \Delta_1 = 1,\ \Delta_2 = 1 - L_1 = 1 - (-G_4H_4) = 1 + G_4H_4,\ \Delta_3 = 1$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

i.e.

$$\frac{C}{R} = \frac{G_1G_2G_3G_4G_5G_6 + G_1G_2G_7G_6(1 + G_4H_4) + G_1G_2G_3G_4G_8}{\Delta}$$

Example 3.18 Find the transfer function of the system whose signal flow graph is shown in Figure 3.79.

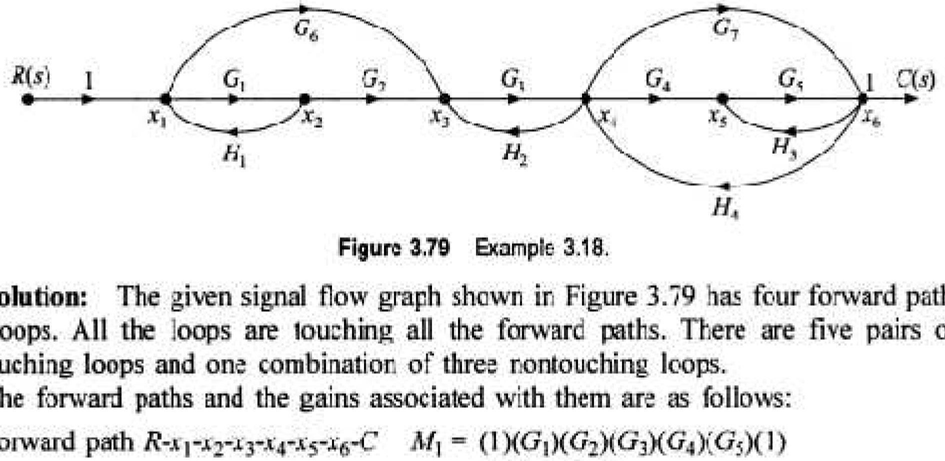


Figure 3.79 Example 3.18.

Solution: The given signal flow graph shown in Figure 3.79 has four forward paths and five loops. All the loops are touching all the forward paths. There are five pairs of two nontouching loops and one combination of three nontouching loops.

The forward paths and the gains associated with them are as follows:

Forward path R-x_1-x_2-x_3-x_4-x_5-x_6-C $\quad M_1 = (1)(G_1)(G_2)(G_3)(G_4)(G_5)(1)$

$= G_1G_2G_3G_4G_5$

Forward path R-x_1-x_3-x_4-x_5-x_6-C $\quad M_2 = (1)(G_6)(G_3)(G_4)(G_5)(1) = G_6G_3G_4G_5$

Forward path R-x_1-x_2-x_3-x_4-x_6-C $\quad M_3 = (1)(G_1)(G_2)(G_3)(G_7)(1) = G_1G_2G_3G_7$

Forward path R-x_1-x_3-x_4-x_6-C $\quad M_4 = (1)(G_6)(G_3)(G_7)(1) = G_6G_3G_7$

The loops and the gains associated with them are as follows:

Loop x_1-x_2-x_1 $\quad L_1 = (G_1)(H_1) = G_1H_1$

Loop x_3-x_4-x_3 $\quad L_2 = (G_3)(H_2) = G_3H_2$

Loop x_5-x_6-x_5 $L_3 = (G_5)(H_3) = G_5H_3$

Loop x_4-x_5-x_6-x_4 $L_4 = (G_4)(G_5)(H_4) = G_4G_5H_4$

Loop x_4-x_6-x_4 $L_5 = (G_7)(H_4) = G_7H_4$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_1-x_2-x_1 and x_3-x_4-x_3 $L_{12} = (G_1H_1)(G_3H_2) = G_1H_1G_3H_2$

Loops x_1-x_2-x_1 and x_5-x_6-x_5 $L_{13} = (G_1H_1)(G_5H_3) = G_1H_1G_5H_3$

Loops x_3-x_4-x_3 and x_5-x_6-x_5 $L_{23} = (G_3H_2)(G_5H_3) = G_3G_5H_2H_3$

Loops x_1-x_2-x_1 and x_4-x_5-x_6-x_4 $L_{14} = (G_1H_1)(G_4G_5H_4) = G_1H_1G_4G_5H_4$

Loops x_1-x_2-x_1 and x_4-x_6-x_4 $L_{15} = (G_1H_1)(G_7H_4) = G_1G_7H_1H_4$

Combinations of three nontouching loops and the gains associated with them are as follows:

Loops L_1, L_2 and L_3 $L_{123} = (G_1H_1)(G_3H_2)(G_5H_3) = G_1H_1G_3H_2G_5H_3$

Since all the loops are touching all the forward paths, $\Delta_1 = 1$, $\Delta_2 = 1$, $\Delta_3 = 1$, and $\Delta_4 = 1$. The determinant of the signal flow graph is given by

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) + (L_{12} + L_{13} + L_{14} + L_{15} + L_{23}) - (L_{123})$$

Therefore, the closed-loop transfer function $\frac{C}{R}$ is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 + M_4\Delta_4}{\Delta} = \frac{G_1G_2G_3G_4G_5 + G_6G_3G_4G_5 + G_1G_2G_3G_7 + G_6G_3G_7}{\Delta}$$

$$= \frac{G_1G_2G_3G_4G_5 + G_6G_3G_4G_5 + G_1G_2G_3G_7 + G_6G_3G_7}{1 - (G_1H_1 + G_3H_2 + G_5H_3 + G_4G_5H_4 + G_7H_4) + (G_1H_1G_3H_2 + G_1H_1G_5H_3 + G_3G_5H_2H_3 + G_1H_1G_4G_5H_4 + G_1G_7H_1H_4) - G_1H_1G_3H_2G_5H_3}$$

Example 3.19 Obtain the transfer function of the system whose signal flow graph is shown in Figure 3.80. Also determine $\frac{x_2}{x_1}, \frac{x_4}{x_1}, \frac{x_7}{x_2}, \frac{x_4}{x_2}$ and $\frac{x_7}{x_4}$.

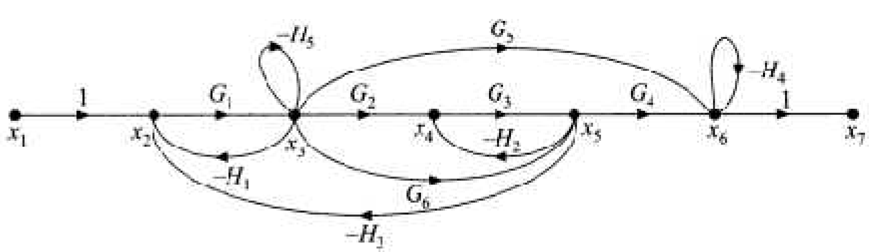


Figure 3.80 Example 3.19.

Solution: The signal flow graph shown in Figure 3.80 has three forward paths and six loops. There are seven pairs of two nontouching loops and two pairs of three nontouching loops. All loops are touching two forward paths and one loop is not touching the third forward path.

The forward paths and the gains associated with them are as follows:

Forward path x_1-x_2-x_3-x_4-x_5-x_6-x_7 $M_1 = (1)(G_1)(G_2)(G_3)(G_4)(1) = G_1G_2G_3G_4$

Forward path x_1-x_2-x_3-x_6-x_7 $M_2 = (1)(G_1)(G_5)(1) = G_1G_5$

Forward path x_1-x_2-x_3-x_5-x_6-x_7 $M_3 = (1)(G_1)(G_6)(G_4)(1) = G_1G_6G_4$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $L_1 = (G_1)(-H_1) = -G_1H_1$

Loop x_3-x_3 $L_2 = -H_5$

Loop x_4-x_5-x_4 $L_3 = (G_3)(-H_2) = -G_3H_2$

Loop x_6-x_6 $L_4 = -H_4$

Loop x_2-x_3-x_4-x_5-x_2 $L_5 = (G_1)(G_2)(G_3)(-H_3) = -G_1G_2G_3H_3$

Loop x_2-x_3-x_5-x_2 $L_6 = (G_1)(G_6)(-H_3) = -G_1G_6H_3$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2 and x_4-x_5-x_4 $L_{13} = (-G_1H_1)(-G_3H_2) = G_1H_1G_3H_2$

Loops x_2-x_3-x_2 and x_6-x_6 $L_{14} = (-G_1H_1)(-H_4) = G_1H_1H_4$

Loops x_3-x_3 and x_4-x_5-x_4 $L_{23} = (-H_5)(-G_3H_2) = G_3H_2H_5$

Loops x_3-x_3 and x_6-x_6 $L_{24} = (-H_5)(-H_4) = H_5H_4$

Loops x_4-x_5-x_4 and x_6-x_6 $L_{34} = (-G_3H_2)(-H_4) = G_3H_2H_4$

Loops x_2-x_3-x_4-x_5-x_2 and x_6-x_6 $L_{45} = (-G_1G_2G_3H_3)(-H_4) = G_1G_2G_3H_3H_4$

Loops x_2-x_3-x_5-x_2 and x_6-x_6 $L_{46} = (-G_1G_6H_3)(-H_4) = G_1G_6H_3H_4$

There are two pairs of three nontouching loops. Those combinations and the products of gains associated with them are as follows:

Loops x_3-x_3, x_4-x_5-x_4, and x_6-x_6 $L_{234} = (-H_5)(-G_3H_2)(-H_4) = -G_3H_2H_4H_5$

Loops x_2-x_3-x_2, x_4-x_5-x_4, and x_6-x_6 $L_{134} = (-G_1H_1)(-G_3H_2)(-H_4) = -G_1H_1G_3H_2H_4$

The values of Δ_1, Δ_2 and Δ_3 are as follows:

$\Delta_1 = 1$, because all loops are touching the first forward path.

$\Delta_2 = 1 + G_3H_2$ because L_3 with a gain of $-G_3H_2$ is not touching the second forward path.

$\Delta_3 = 1$, because all loops are touching the third forward path.

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + (L_{13} + L_{14} + L_{23} + L_{24} + L_{34} + L_{45} + L_{46}) - (L_{134} + L_{234})$$

$$\therefore \quad \Delta = 1 - (-G_1H_1 - H_5 - G_3H_2 - H_4 - G_1G_2G_3H_3 - G_1G_6H_3) + (G_1G_3H_1H_2 + G_1H_1H_4 + H_4H_5 + G_3H_2H_5 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_6H_3H_4) - (-G_1G_3H_1H_2H_4 - G_3H_2H_4H_5)$$

The transfer function

$$T = \frac{x_7}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3}{\Delta}$$

$$\therefore \quad \frac{x_7}{x_1} = \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2) + G_1G_4G_6}{\begin{array}{l}1+G_1H_1 + H_5 + G_3H_2 + H_4 + G_1G_2G_3H_3 + G_1G_6H_3 + G_1G_3H_1H_2 \\ + G_1H_1H_4 - H_4H_5 + G_3H_2H_5 + G_3H_2H_4 + G_1G_2G_3H_3H_4 + G_1G_6H_3H_4 \\ + G_1G_3H_1H_2H_4 + G_3H_2H_4H_5\end{array}}$$

Determination of $\frac{x_2}{x_1}$

To determine $\frac{x_2}{x_1}$, there is only one forward path between x_1 and x_2 and the gain of that forward path is unity. Loops 2, 3 and 4 are not touching this forward path. Combinations of two nontouching loops which are not touching this forward path are L_{23}, L_{34} and L_{24}. One combination of three nontouching loops which are not touching this forward path is L_{234}. Therefore

$$\frac{x_2}{x_1} = \frac{M_1\Delta_1}{\Delta}$$

where $M_1 = 1$

and $\Delta_1 = 1 - (-H_5 - G_3H_2 - H_4) + (G_3H_2H_5 + G_3H_2H_4 + H_4H_5) - (-G_3H_2H_4H_5)$

$= 1 + H_5 + H_4 + G_3H_2 + G_3H_2H_5 + G_3H_2H_4 + H_4H_5 + G_3H_2H_4H_5$

$$\therefore \quad \frac{x_2}{x_1} = \frac{1 + H_5 + H_4 + G_3H_2 + G_3H_2H_5 + G_3H_2H_4 + H_4H_5 + G_3H_2H_4H_5}{\Delta}$$

Determination of $\frac{x_4}{x_1}$

To determine $\frac{x_4}{x_1}$, there are two forward paths between x_1 and x_4. Only one loop L_4 is not touching both these two forward paths.

Forward path x_1-x_2-x_3-x_4 $\quad M_1 = G_1G_2 \quad \Delta_1 = 1 - (-H_4) = 1 + H_4$

Forward path x_1-x_2-x_3-x_5-x_4 $\quad M_2 = -G_1G_6H_2 \quad \Delta_2 = 1 - (-H_4) = 1 + H_4$

$$\therefore \quad \frac{x_4}{x_1} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{(G_1G_2 - G_1G_6H_2)(1 + H_4)}{\Delta}$$

Determination of $\frac{x_7}{x_2}$

$$\frac{x_7}{x_2} = \frac{\frac{x_7}{x_1}}{\frac{x_2}{x_1}} = \frac{\dfrac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2) + G_1G_4G_6}{\Delta}}{\dfrac{1 + H_5 + H_4 + G_3H_2 + G_3H_2H_5 + G_3H_2H_4 + H_4H_5 + G_3H_2H_4H_5}{\Delta}}$$

i.e. $$\frac{x_7}{x_2} = \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2) + G_1G_4G_6}{1+H_5+H_4+G_3H_2+G_3H_2H_5+G_3H_2H_4+H_4H_5+G_3H_2H_4H_5}$$

Determination of $\dfrac{x_7}{x_4}$

$$\frac{x_7}{x_4} = \frac{\dfrac{x_7}{x_1}}{\dfrac{x_4}{x_1}} = \frac{\dfrac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2) + G_1G_4G_6}{\Delta}}{\dfrac{(G_1G_2 - G_1G_6H_2)(1+H_4)}{\Delta}}$$

i.e. $$\frac{x_7}{x_4} = \frac{G_1G_2G_3G_4 + G_1G_5(1+G_3H_2) + G_1G_4G_6}{(G_1G_2 - G_1G_6H_2)(1+H_4)}$$

Determination of $\dfrac{x_4}{x_2}$

$$\frac{x_4}{x_2} = \frac{\dfrac{x_4}{x_1}}{\dfrac{x_2}{x_1}} = \frac{(G_1G_2 - G_1G_6H_2)(1+H_4)}{1+H_5+H_4+G_3H_2+G_3H_2H_5+G_3H_2H_4+H_4H_5+G_3H_2H_4H_5}$$

Example 3.20 Obtain the overall transfer function *C/R* from the signal flow graph shown in Figure 3.81.

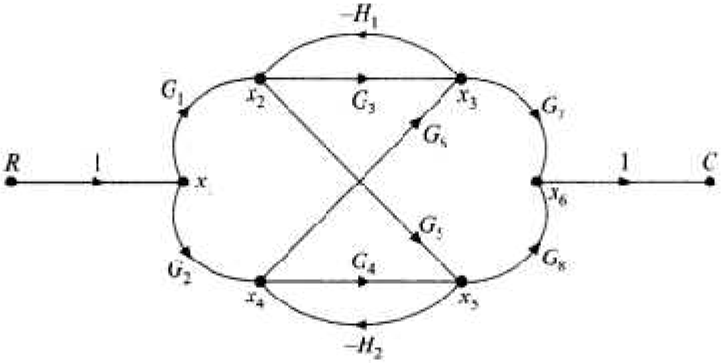


Figure 3.81 Example 3.20.

Solution: For the signal flow graph shown in Figure 3.81, there are six forward paths, three loops and one pair of two nontouching loops. One loop is not touching one forward path and another loop is not touching another forward path.

The forward paths and the gains associated with them are as follows:

Forward path R-x_1-x_2-x_3-x_6-C $\quad M_1 = (1)(G_1)(G_3)(G_7)(1) = G_1G_3G_7$

Forward path R-x_1-x_2-x_5-x_6-C $\quad M_2 = (1)(G_1)(G_5)(G_8)(1) = G_1G_5G_8$

Forward path R-x_1-x_2-x_5-x_4-x_3-x_6-C $\quad M_3 = (1)(G_1)(G_5)(-H_2)(G_6)(G_7)(1)$
$= -G_1G_5H_2G_6G_7$

Forward path R-x_1-x_4-x_5-x_6-C $\quad M_4 = (1)(G_2)(G_4)(G_8)(1) = G_2G_4G_8$

Forward path R-x_1-x_4-x_3-x_6-C $\quad M_5 = (1)(G_2)(G_6)(G_7)(1) = G_2G_6G_7$

Forward path R-x_1-x_4-x_3-x_2-x_5-x_6-C $\quad M_6 = (1)(G_2)(G_6)(-H_1)(G_5)(G_8)(1)$
$= -G_2G_5G_6G_8H_1$

The loops and the gains associated with them are as follows:

Loop x_2-x_3-x_2 $\quad L_1 = (G_3)(-H_1) = -G_3H_1$

Loop x_4-x_5-x_4 $\quad L_2 = (G_4)(-H_2) = -G_4H_2$

Loop x_2-x_5-x_4-x_3-x_2 $\quad L_3 = (G_5)(-H_2)(G_6)(-H_1) = G_5G_6H_1H_2$

The pairs of two nontouching loops and the products of gains associated with them are as follows:

Loops x_2-x_3-x_2 and x_4-x_5-x_4 $\quad L_{12} = (-G_3H_1)(-G_4H_2) = G_3G_4H_1H_2$

Loop 2 is not touching the first forward path. Therefore,

$$\Delta_1 = 1 - (-G_4H_2) = 1 + G_4H_2$$

Loop 1 is not touching the fourth forward path. Therefore,

$$\Delta_4 = 1 - (-G_3H_1) = 1 + G_3H_1$$

All the loops are touching all other forward paths. Therefore

$$\Delta_2 = 1 = \Delta_3 = \Delta_5 = \Delta_6$$

The determinant of the signal flow graph is

$$\Delta = 1 - (L_1 + L_2 + L_3) + (L_{12})$$
$$= 1 - (-G_3H_1 - G_4H_2 + G_5G_6H_1H_2) + G_3G_4H_1H_2$$

Applying Mason's gain formula, the transfer function is

$$\frac{C}{R} = \frac{M_1\Delta_1 + M_2\Delta_2 + M_3\Delta_3 + M_4\Delta_4 + M_5\Delta_5 + M_6\Delta_6}{\Delta}$$

$$= \frac{G_1G_3G_7(1+G_4H_2) + G_1G_5G_8 - G_1G_5G_6G_7H_2 + G_2G_4G_8(1+G_3H_1) + G_2G_6G_7 - G_2G_5G_6G_8H_1}{1+G_3H_1 + G_4H_2 - G_5G_6H_1H_2 + G_3G_4H_1H_2}$$

Example 3.21 Figure 3.82 shows the signal flow graph of a system with two inputs and two outputs. Find expressions for outputs C_1 and C_2. Also determine the condition that makes C_1 independent of R_2, and C_2 independent of R_1.

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The steady-state is reached mathematically only after an infinite time. In particular, however, a reasonable estimate of the response time is the length of time the response curve needs to reach and stay within the 2% line of the final value, or four time constants.

4.3.2 Unit-Ramp Response of First-Order Systems

The output response of a first-order system for the unit-ramp input ($R(s) = 1/s^2$) is given by

$$C(s) = \frac{1}{s^2} \cdot \frac{1}{Ts+1} \tag{4.4}$$

Expanding Eq. (4.4) into partial fractions, we have

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+1/T} \tag{4.5}$$

Taking the inverse Laplace transform of Eq. (4.5), we get

$$c(t) = t - T + Te^{-t/T} = t - T\,(1 - e^{-t/T}) \text{ for } t \geq 0 \tag{4.6}$$

The error signal is then

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= T(1 - e^{-t/T}) \end{aligned}$$

and the steady-state error is given by

$$e_{ss} = \underset{t \to \infty}{\text{Lt}}\; e(t) = T$$

Thus, the first-order system under consideration will track the unit-ramp input with a steady-state error T, which is equal to the time constant of the system as shown in Figure 4.5. The smaller the time constant T, the smaller the steady-state error in following the ramp input. Reducing the time constant therefore not only improves its speed of response but also reduces its steady-state error to a ramp input.

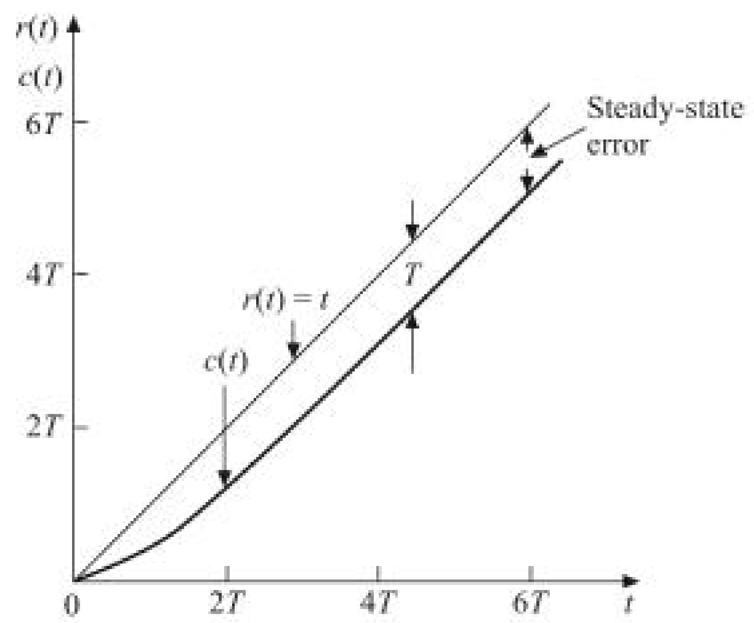

Figure 4.5 Unit-ramp response of the first-order system.

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For the unit-step input, which is the derivative of the unit-ramp input, the output $c(t)$ is

$$c(t) = 1 - e^{-t/T}, \quad \text{for} \quad t \geq 0$$

Finally, for the unit-impulse input, which is the derivative of the unit-step input, the output $c(t)$ is

$$c(t) = \frac{1}{T} e^{-t/T}, \quad \text{for} \quad t \geq 0$$

Comparison of the system responses to these three inputs clearly indicates that the response of the derivative of an input signal can be obtained by differentiating the response of the system to the original signal. It can also be seen that the response to the integral of the original signal can be obtained by integrating the response of the system to the original signal and by determining the integration constant from the zero output initial conditions. This is a property of linear time-invariant systems. Linear time-varying systems and nonlinear systems do not possess this property.

4.4 SECOND-ORDER SYSTEMS

4.4.1 Response of Second-Order System to the Unit-Step Input

Consider the second-order system shown in Figure 4.7. The closed-loop transfer function $C(s)/R(s)$ of the system is given by

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \tag{4.9}$$

where ξ = damping ratio (or damping factor)
and ω_n = undamped natural frequency.

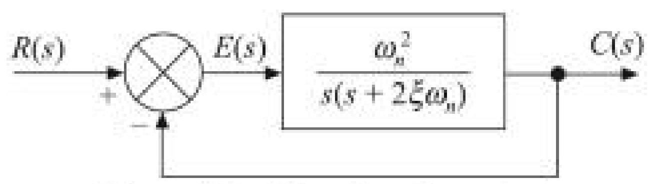

Figure 4.7 Second-order system.

This form is called the standard form of the second-order system. The dynamic behaviour of the second-order system can then be described in terms of two parameters ξ and ω_n.

If $\xi = 0$, the poles are purely imaginary and lie on the $j\omega$-axis. The system is then called undamped. The transient response does not die out. It is purely oscillatory. If $0 < \xi < 1$, the closed-loop poles are complex conjugates and lie in the left half of the s-plane. The system is then called underdamped, and the transient response is oscillatory. If $\xi = 1$, the poles are real, negative and equal. The system is called critically damped. The response rises slowly and reaches the final value. If $\xi > 1$, the poles are real, negative and unequal. The system is called overdamped. The output rises towards its final value slowly. Critically-damped and overdamped systems do not exhibit any overshoot.

The time response of any system is characterized by the roots of the denominator polynomial $q(s)$, which in fact are the poles of the transfer function.

The denominator polynomial $q(s)$ is therefore called the characteristic polynomial and

$$q(s) = 0$$

is called the *characteristic equation*. The characteristic equation of the system under consideration is

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

The roots of this characteristic equation are given by

$$s^2 + 2\xi\omega_n s + \omega_n^2 = (s - s_1)(s - s_2)$$

For $\xi < 1$,

$$s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2} = -\xi\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$, is called the *damped natural frequency*.

Most control systems with the exception of robotic control systems are designed with damping factor $\xi < 1$, to have high response speed.

Response of an underdamped system $(0 < \xi < 1)$: In this case, $C(s)/R(s)$ can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s + \xi\omega_n + j\omega_d)(s + \xi\omega_n - j\omega_d)} \tag{4.10}$$

For a unit-step input, $R(s) = 1/s$. Therefore, Eq. (4.10) becomes

$$\begin{aligned}
C(s) &= \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} \\
&= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{1}{s} - \frac{s + 2\xi\omega_n}{[(s+\xi\omega_n)^2 + \omega_n^2 - \omega_n^2\xi^2]} \\
&= \frac{1}{s} - \frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2} \\
&= \frac{1}{s} - \frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi}{\sqrt{1-\xi^2}}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2 + \omega_d^2}
\end{aligned} \tag{4.11}$$

Taking the inverse Laplace transform of Eq. (4.11),

$$\begin{aligned}
c(t) &= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\omega_d t \\
&= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right)
\end{aligned}$$

It is damped sinusoid. The response reaches a steady-state value of $c_{ss} = 1$, i.e. the steady-state error of this system approaches zero. The time response for various values of ξ plotted against normalized time $\omega_n t$ is shown in Figure 4.9. The system breaks into continuous oscillations for $\xi = 0$.

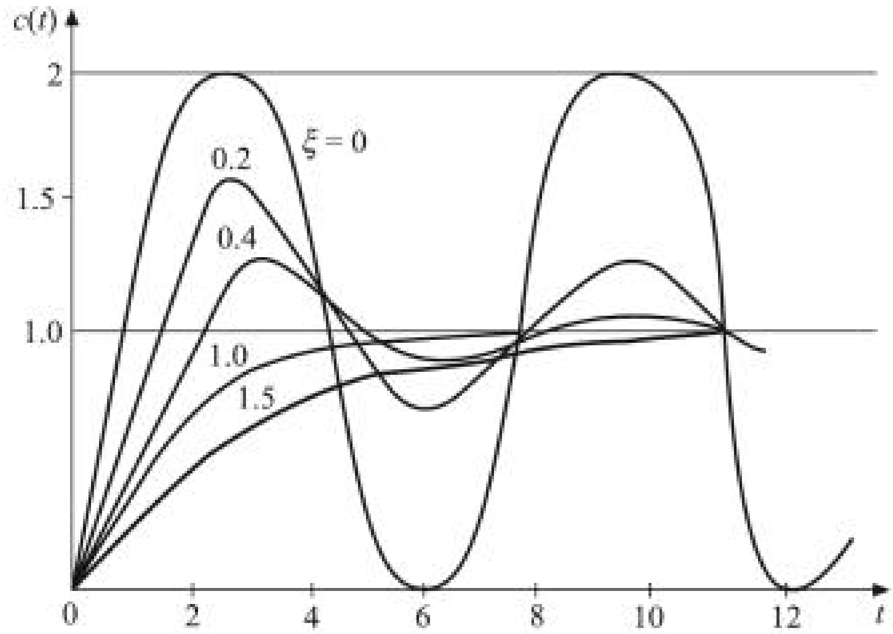

Figure 4.9 Unit-step response curves of second-order systems.

As ξ is increased, the response becomes progressively less oscillatory till it becomes critically damped (just non-oscillatory) for $\xi = 1$, and becomes overdamped for $\xi > 1$.

Figure 4.10 shows the locus of the poles of the second-order system discussed above with ω_n held constant and ξ varying from 0 to ∞. As ξ increases, the poles move away from the imaginary axis along a circular path of radius ω_n meeting at the point $\sigma = -\omega_n$ and then

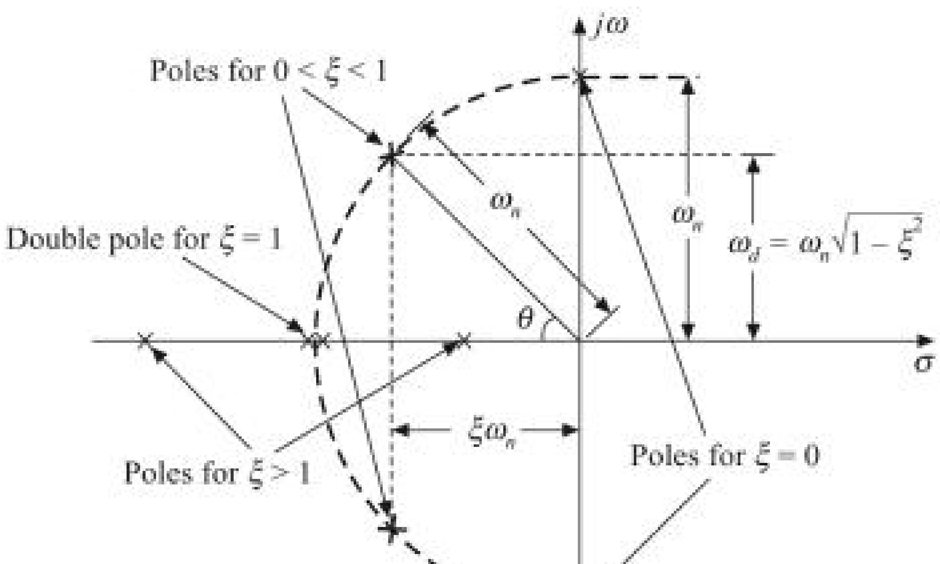

Figure 4.10 Pole locations for a second-order system.

The relation between θ and ξ is shown in Figure 4.12.

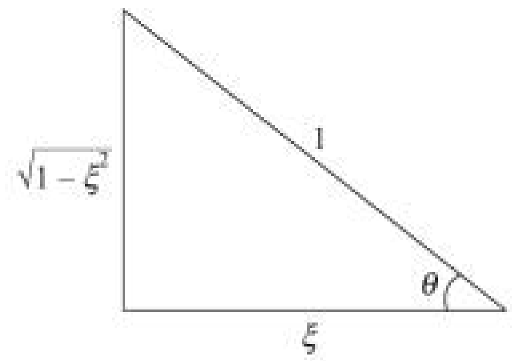

Figure 4.12 Relation between θ and ξ.

The first undershoot occurs at $t = \dfrac{2\pi}{\omega_d}$

The second overshoot occurs at $t = \dfrac{3\pi}{\omega_d}$, and so on.

Peak overshoot M_p: We know that the output of a second-order underdamped system excited by a unit-step input is given by

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

The peak overshoot is the difference between the peak value and the reference input. Therefore,

$$M_p = c(t_p) - 1 = \left[1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)\right] - 1$$

$$= -\frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}} \sin(\omega_d t_p + \theta)$$

Substituting Eq. (4.15), we get

$$M_p = -\frac{e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}}{\sqrt{1-\xi^2}} \sin\left(\omega_d \frac{\pi}{\omega_d} + \theta\right)$$

$$= -\frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}(-\sin\theta)$$

$$= \frac{e^{-\pi\xi/\sqrt{1-\xi^2}}}{\sqrt{1-\xi^2}}\sqrt{1-\xi^2} = e^{-\pi\xi/\sqrt{1-\xi^2}} \tag{4.16}$$

Therefore, the peak percent overshoot is

$$100 \times e^{-\pi\xi/\sqrt{1-\xi^2}}\ \%$$

The poles that are close to the imaginary axis in the left half s-plane give rise to transient responses that will decay relatively slowly and are called *dominant poles*, whereas the poles that are far away from the imaginary axis (relative to the dominant poles) correspond to fast-decaying time responses and are called *insignificant poles*. In practice, if the magnitude of the real part of a pole is at least 5 to 10 times that of a dominant pole or a pair of complex dominant poles, then the pole may be regarded as insignificant in so far as the transient response is concerned. The roots that are closer to the imaginary axis will dominate the transient response, and these are defined as the dominant roots of the characteristic equation of the system.

4.8.1 The Relative Damping Ratio

When a system is higher than the second order, we can no longer strictly use the damping ratio ξ and the undamped natural frequency ω_n, which are defined for the prototype second-order systems. However, if the system dynamics can be accurately represented by a pair of complex conjugate dominant poles, then we can still use ξ and ω_n to indicate the dynamics of the transient response and the damping ratio in this case is referred to as the relative damping ratio of the system.

For example, consider the following closed-loop transfer function:

$$M(s) = \frac{C(s)}{R(s)} = \frac{20}{(s+10)(s^2+2s+2)}$$

The pole at $s = -10$ is ten times the real part of the complex conjugate poles, which are at $-1 \pm j1$. We can refer to the relative damping ratio of the system as 0.707.

4.8.2 The Proper Way of Neglecting Insignificant Poles with Consideration of Steady-State Response

If the insignificant poles are neglected as they are, the transient response is unchanged, but the steady-state response gets affected.

For example, consider the following closed-loop transfer function:

$$M(s) = \frac{C(s)}{R(s)} = \frac{20}{(s+10)(s^2+2s+2)}$$

Neglecting the pole far away from the origin, the third-order system can be approximated by a second-order system with the transfer function

$$\frac{C(s)}{R(s)} = \frac{20}{(s^2+2s+2)}$$

The transient response of the above two transfer functions will be the same but their steady-state responses will be different. To maintain the same steady-state response, the transfer function can be written as

$$\frac{C(s)}{R(s)} = \frac{20}{10(s/10+1)(s^2+2s+2)}$$

The term $s/10$ can be neglected compared to 1. So the third-order system can be written in terms of a second-order system as

$$\frac{C(s)}{R(s)} = \frac{20}{10(s^2 + 2s + 2)}$$

Both the above systems will have the same steady-state error.

Example 4.1 For the system with the following transfer function, determine the type and order of the system:

(a) $G(s)H(s) = \dfrac{K(s+3)}{s^2(s+2)(s+5)}$ (b) $G(s)H(s) = \dfrac{K}{s(s+1)(s^2+2s+3)}$

(c) $G(s)H(s) = \dfrac{s+2}{(s-3)(s+0.1)}$ (d) $G(s)H(s) = \dfrac{20}{s^3(s^4+8s^2+16)}$

Solution:

(a) The open-loop transfer function has two poles at the origin of the s-plane, so it is a type-2 system. The highest power of s present in the denominator is 4, so it is a fourth-order system.

(b) The open-loop transfer function has one pole at the origin of the s-plane, so it is a type-1 system. The highest power of s present in the denominator is 4, so it is a fourth-order system.

(c) The open-loop transfer function has no pole at the origin of the s-plane, so it is type-0 system. The highest power of s present in the denominator is 2, so it is a second-order system.

(d) The open-loop transfer function has three poles at the origin of the s-plane, so it is a type-3 system. The highest power of s present in the denominator is 7, so it is a seventh-order system.

Example 4.2 The closed-loop transfer functions of certain second-order unity feedback control systems are given below. Determine the type of damping in the systems.

(a) $\dfrac{C(s)}{R(s)} = \dfrac{8}{s^2+3s+8}$ (b) $\dfrac{C(s)}{R(s)} = \dfrac{2}{s^2+4s+2}$

(c) $\dfrac{C(s)}{R(s)} = \dfrac{2}{s^2+2s+1}$ (d) $\dfrac{C(s)}{R(s)} = \dfrac{2}{s^2+4}$

Solution: Comparing the given transfer functions with the standard form of the transfer function of a second-order system, i.e. with

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Example 4.3 A first-order system and its response to a unit-step input are shown in Figure 4.15. Determine the system parameters a and K.

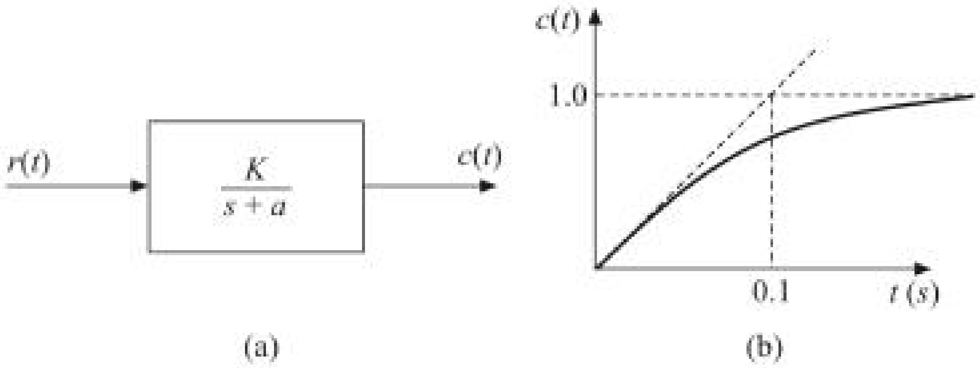

Figure 4.15 Example 4.3: (a) system and (b) response.

Solution: The system transfer function is

$$M(s) = \frac{C(s)}{R(s)} = \frac{K}{s+a}$$

For a unit-step input,

$$R(s) = \frac{1}{s}$$

Therefore, the output

$$C(s) = R(s)M(s) = \frac{1}{s} \cdot \frac{K}{s+a}$$

The steady-state value of the output is

$$\underset{t\to\infty}{\text{Lt}}\ c(t) = \underset{t\to\infty}{\text{Lt}}\ L^{-1}\left[\frac{K}{s(s+a)}\right] = \frac{K}{a} \underset{t\to\infty}{\text{Lt}}\ L^{-1}\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

$$= \frac{K}{a} \underset{t\to\infty}{\text{Lt}}\ (1 - e^{-at}) = \frac{K}{a}$$

From Figure 4.15(b), $c(\infty) = 1$. Therefore,

$$\frac{K}{a} = 1$$

or

$$K = a$$

The initial slope of the output $= \underset{t\to 0}{\text{Lt}}\left[\frac{d}{dt}c(t)\right] = \frac{1}{0.1} = 10$

$$L\left[\frac{d}{dt}c(t)\right] = sC(s) = s\left[\frac{K}{s(s+a)}\right] = \frac{K}{s+a}$$

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Taking the natural logarithm on both sides,

$$\ln M_p = \frac{-\pi\xi}{\sqrt{1-\xi^2}}$$

Squaring both sides

$$(\ln M_p)^2 = \frac{\pi^2\xi^2}{1-\xi^2}$$

i.e.

$$(1-\xi^2)(\ln M_p)^2 = \pi^2\xi^2$$

i.e.

$$(\ln M_p)^2 = \xi^2[\pi^2 + (\ln M_p)^2]$$

or

$$\xi^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2}$$

Equating both the expressions for ξ^2,

$$\xi^2 = \frac{(\ln M_p)^2}{\pi^2 + (\ln M_p)^2} = \frac{1}{4KT}$$

i.e.

$$K = \frac{\pi^2 + (\ln M_p)^2}{4T(\ln M_p)^2}$$

When $K = K_1$, $M_p = 0.75$. Therefore,

$$K_1 = \frac{\pi^2 + (\ln 0.75)^2}{4T(\ln 0.75)^2} = \frac{9.952}{0.331T} = \frac{30.06}{T}$$

When $K = K_2$, $M_p = 0.25$. Therefore,

$$K_2 = \frac{\pi^2 + (\ln 0.25)^2}{4T(\ln 0.25)^2} = \frac{11.79}{7.68T} = \frac{1.53}{T}$$

$\therefore$

$$\frac{K_1}{K_2} = \frac{(1/T)\times 30.06}{(1/T)\times 1.53} = 19.6$$

i.e.

$$K_1 = 19.6\ K_2$$

or

$$K_2 = \frac{1}{19.6}K_1$$

This indicates that, to reduce the peak overshoot from 0.75 to 0.25, K should be reduced by 19.6 times.

Example 4.14 A unity feedback control system has an open-loop transfer function

$$G(s) = \frac{5}{s(s+1)}$$

Find the rise time, percentage overshoot, peak time and settling time for a step input of 10 units. Also determine the peak overshoot.

Solution: Note that the formulae for the rise time, percentage overshoot, peak time and settling time remain the same for unit-step input and step input of any amplitude. Only the peak overshoot varies. The peak overshoot for a step input of 10 units is 10 times the peak overshoot for a unit-step input.

The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{5}{s(s+1)}}{1+\frac{5}{s(s+1)}} = \frac{5}{s^2+s+5}$$

Comparing this transfer function with the standard form of the transfer function of a second-order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{5}{s^2+s+5}$$

$\therefore$ $$\omega_n^2 = 5$$

or $$\omega_n = \sqrt{5} = 2.236 \text{ rad/s}$$

$$2\xi\omega_n = 1$$

$\therefore$ $$\xi = \frac{1}{2\omega_n} = \frac{1}{2\times 2.236} = 0.223$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 2.236\sqrt{1-0.223^2} = 2.124 \text{ rad/s}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} = \tan^{-1}\frac{\sqrt{1-0.223^2}}{0.223} = 1.346 \text{ rad/s}$$

The rise time

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{3.141-1.346}{2.124} = 0.845\,\text{s}$$

The percentage overshoot

$$\%M_p = e^{(-\pi\xi/\sqrt{1-\xi^2})}\times 100\%$$

$$= e^{(-\pi \times 0.223/\sqrt{1-0.223^2})} \times 100\%$$

$$= 0.478 \times 100 = 47.8\%$$

The peak overshoot for a unit-step input is

$$\frac{47.8}{100} = 0.478$$

For an input of 10 units, the peak overshoot is

$$0.478 \times 10 = 4.78$$

The peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.124} = 1.479\,\text{s}$$

The time constant

$$T = \frac{1}{\xi\omega_n} = \frac{1}{0.223 \times 2.236} = 2\,\text{s}$$

For 5% error, the settling time

$$t_s = 3T = 3 \times 2 = 6\text{s}$$

For 2% error, the settling time

$$t_s = 4T = 4 \times 2 = 8\text{s}$$

Example 4.15 A unity feedback control system is characterized by the following open-loop transfer function

$$G(s) = \frac{0.4s + 1}{s(s + 0.6)}$$

Determine its transient response for a unit-step input. Evaluate the maximum overshoot and the corresponding peak time.

Solution: The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{\dfrac{0.4s + 1}{s(s + 0.6)}}{1 + \dfrac{0.4s + 1}{s(s + 0.6)}}$$

$$= \frac{0.4s + 1}{s^2 + s + 1}$$

For a unit-step input, $r(t) = 1$. Therefore,

$$R(s) = \frac{1}{s}$$

The output of the system is therefore given by

$$C(s) = R(s) \times \frac{G(s)}{1+G(s)} = \frac{1}{s} \times \frac{0.4s+1}{s^2+s+1}$$

$$\therefore \quad C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+s+1} = \frac{1}{s} - \frac{s+0.6}{s^2+s+1}$$

$$= \frac{1}{s} - \frac{s+0.5}{s^2+s+0.25+0.75} - \frac{0.1}{s^2+s+0.25+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+(\sqrt{0.75})^2} - \frac{0.1}{(\sqrt{0.75})}\frac{(\sqrt{0.75})}{(s+0.5)^2+(\sqrt{0.75})^2}$$

Taking the inverse Laplace transform on both sides

$$c(t) = 1 - e^{-0.5t}\left[\cos\sqrt{0.75}t - \frac{0.1}{\sqrt{0.75}}\sin\sqrt{0.75}t\right]$$

This value of $c(t)$ gives the total time response of the system.

The transient response is that part of the time response which tends to zero as t tends to infinity. Here in the expression for $c(t)$, as $t \to \infty$, the exponential component $e^{-0.5t}$ tends to zero. Hence the transient response is given by the damped sinusoidal component.

Hence the transient response is

$$c(t) = e^{-0.5t}\left[\cos\sqrt{0.75}t - \frac{0.1}{\sqrt{0.75}}\sin\sqrt{0.75}t\right]$$

The values of ξ and ω_n can be estimated by comparing the characteristic equation of the system transfer function with the standard form of the second-order characteristic equation. Therefore,

$$s^2 + 2\xi\omega_n s + \omega_n^2 = s^2 + s + 1$$

$$\omega_n^2 = 1$$

or $$\omega_n = 1$$

$$2\xi\omega_n = 1$$

$$\therefore \quad \xi = \frac{1}{2\omega_n} = \frac{1}{2\times 1} = 0.5$$

The maximum overshoot

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{-3.147\times 0.5/\sqrt{1-0.5^2}} = 0.163$$

The % maximum overshoot is

$$M_p \times 100 = 0.163 \times 100 = 16.3\%$$

The peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{1\times\sqrt{1-0.5^2}} = 3.628 \text{ s}$$

Example 4.16 The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{K}{s(1+sT)}$$

(a) By what factor the amplifier gain K should be multiplied so that the damping ratio is increased from 0.2 to 0.8?

(b) By what factor the time constant T should be multiplied so that the damping ratio is reduced from 0.9 to 0.3?

Solution: The closed-loop transfer function of the system is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K}{s(1+sT)}}{1+\dfrac{K}{s(1+sT)}} = \frac{\dfrac{K}{T}}{s^2+\dfrac{s}{T}+\dfrac{K}{T}}$$

Comparing the characteristic equation $s^2+\frac{s}{T}+\frac{K}{T}=0$, with the standard form of the characteristic equation $s^2+2\xi\omega_n s+\omega_n^2=0$ of a second-order system.

$$2\xi\omega_n = \frac{1}{T}$$

and $$\omega_n^2 = \frac{K}{T}$$

i.e. $$\omega_n = \sqrt{\frac{K}{T}}$$

$\therefore$ $$2\xi\sqrt{\frac{K}{T}} = \frac{1}{T}$$

or $$\xi = \frac{1}{2\sqrt{KT}}$$

(a) When $\xi = \xi_1 = 0.2$, let $K = K_1$.
When $\xi = \xi_2 = 0.8$, let $K = K_2$.

$\therefore$ $$\frac{\xi_1}{\xi_2} = \frac{0.2}{0.8} = \frac{1}{4} = \frac{1}{2\sqrt{K_1T}}\times 2\sqrt{K_2T} = \sqrt{\frac{K_2}{K_1}}$$

i.e.
$$\frac{K_2}{K_1} = \left(\frac{\xi_1}{\xi_2}\right)^2 = \frac{1}{16}$$

or
$$K_2 = \frac{1}{16} K_1$$

Hence the gain K_1, at which $\xi = 0.2$ should be multiplied by 1/16 to increase the damping ratio from 0.2 to 0.8.

(b) When $\xi = \xi_1 = 0.9$, let $T = T_1$.
When $\xi = \xi_2 = 0.3$, let $T = T_2$.

$$\frac{\xi_1}{\xi_2} = \frac{0.9}{0.3} = 3 = \frac{1}{2\sqrt{KT_1}} \times 2\sqrt{KT_2} = \sqrt{\frac{T_2}{T_1}}$$

$\therefore$
$$\frac{T_2}{T_1} = \left(\frac{\xi_1}{\xi_2}\right)^2 = 9$$

or
$$T_2 = 9T_1$$

Hence the original time constant T_1 should be multiplied by 9 to reduce the damping ratio from 0.9 to 0.3.

Example 4.17 A unity feedback control system has an amplifier with gain $K_A = 16$ and gain ratio,

$$G(s) = \frac{1}{s(s+4)}$$

in the forward path. A derivative feedback $H(s) = sK_0$ is introduced as a minor loop around $G(s)$. Determine the derivative feedback constant K_0 so that the system damping factor is 0.5.

Solution: The given system can be represented by the block diagram shown in Figure 4.20.

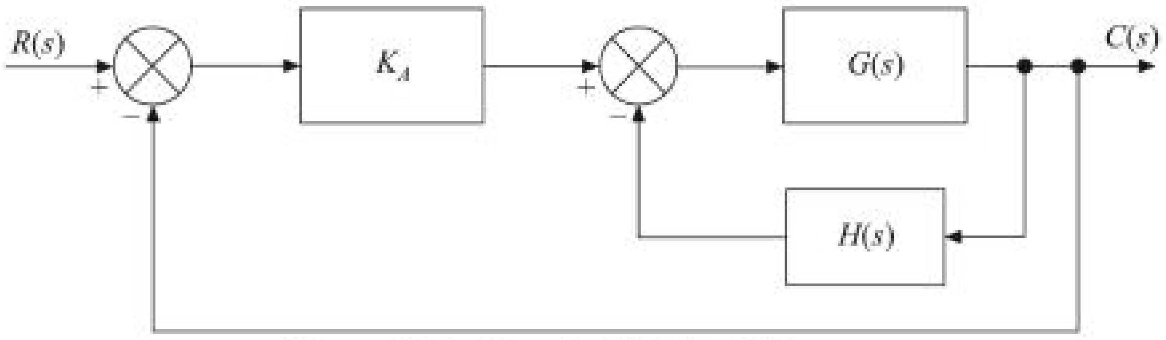

Figure 4.20 Example 4.17: Block diagram.

Here $K_A = 16$, $G(s) = \dfrac{1}{s(s+3)}$ and $H(s) = sK_0$.

The closed-loop transfer function can be obtained by using the signal flow graph method as shown in Figure 4.21.

Solution: The block diagram shown in Figure 4.22 can be simplified as shown in Figure 4.23.

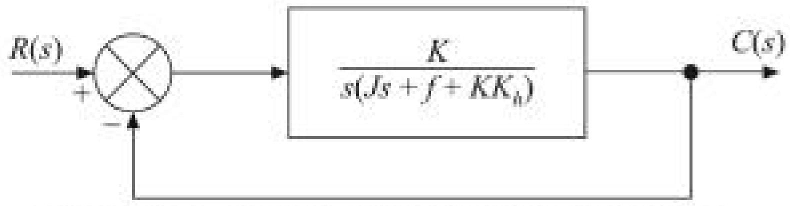

Figure 4.23 Example 4.18: Simplified block diagram.

The transfer function of the system is given by

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (f + KK_h)s + K}$$

$$= \frac{\frac{K}{J}}{s^2 + \frac{f + KK_h}{J} + \frac{K}{J}}$$

Comparing it with the transfer function of a standard second-order system,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

and $$2\xi\omega_n = \frac{f + KK_h}{J}$$

$\therefore$ $$\xi = \frac{f + KK_h}{2\sqrt{\frac{K}{J}}\,.\,J} = \frac{f + KK_h}{2\sqrt{KJ}}$$

Determination of the values of K and K_h

The maximum overshoot M_p is given

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

This value must be 0.2. Thus

$$e^{-\pi\xi/\sqrt{1-\xi^2}} = 0.2 = e^{-1.61}$$

$\therefore$ $$\frac{\pi\xi}{\sqrt{1-\xi^2}} = 1.61$$

Solution:

(a) $G(s) = \dfrac{20}{(s+1)(s+4)}$

The open-loop transfer function does not have any pole at the origin. So it is a type-0 system. Therefore, only a step signal gives rise to a constant steady-state error. Hence

$$K_p = \underset{s\to 0}{\text{Lt}}\ G(s) = \underset{s\to 0}{\text{Lt}}\ \frac{20}{(s+1)(s+4)} = \frac{20}{4} = 5,\ K_v = 0 \text{ and } K_a = 0$$

Therefore, the steady-state error for unit-step input is

$$e_{ss}(t) = \frac{1}{1+K_p} = \frac{1}{1+5} = \frac{1}{6}$$

(b) $G(s) = \dfrac{10(s+4)}{s(s+1)(s+2)}$

The open-loop transfer function has got one pole at the origin of the s-plane. So, it is a type-1 system. Therefore, only a ramp signal gives rise to a constant steady-state error. Hence

$$K_p = \infty,\ K_a = 0$$

and

$$K_v = \underset{s\to 0}{\text{Lt}}\ sG(s) = \underset{s\to 0}{\text{Lt}}\ s \times \frac{10(s+4)}{s(s+1)(s+2)} = 20$$

Therefore, the steady-state error for a unit-ramp input is

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{20} = 0.05$$

(c) $G(s) = \dfrac{20}{s^2(s+1)(s+4)}$

The open-loop transfer function has got two poles at the origin of the s-plane. So it is a type-2 system. Therefore, only a parabolic signal gives rise to a constant steady-state error. Hence

$$K_p = \infty,\ K_v = \infty$$

and

$$K_a = \underset{s\to 0}{\text{Lt}}\ s^2 G(s) = \underset{s\to 0}{\text{Lt}}\ s^2 \times \frac{20}{s^2(s+1)(s+4)} = 5$$

Therefore, the steady-state error for a unit-parabolic input is

$$e_{ss}(t) = \frac{1}{K_a} = \frac{1}{5} = 0.2$$

Example 4.23 A unity feedback system is characterized by the open-loop transfer function

$$G(s) = \frac{1}{s(0.5s+1)(0.2s+1)}$$

Determine the steady-state errors for unit-step, unit-ramp and unit-acceleration input.

Solution: For a unity feedback system, $H(s) = 1$.

The position error constant

$$K_p = \operatorname*{Lt}_{s \to 0} G(s)$$

$$= \operatorname*{Lt}_{s \to 0} \frac{1}{s(0.5s+1)(0.2s+1)} = \infty$$

Therefore, the steady-state error for a unit-step input is

$$e_{ss}(t) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

The velocity error constant

$$K_v = \operatorname*{Lt}_{s \to 0} sG(s)$$

$$= \operatorname*{Lt}_{s \to 0} s \times \frac{1}{s(0.5s+1)(0.2s+1)} = 1$$

Therefore, the steady-state error for a unit-ramp input is

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{1} = 1$$

The acceleration error constant

$$K_a = \operatorname*{Lt}_{s \to 0} s^2 G(s)$$

$$= \operatorname*{Lt}_{s \to 0} s^2 \times \frac{1}{s(0.5s+1)(0.2s+1)} = 0$$

Therefore, the steady-state error for a unit-acceleration input is

$$e_{ss}(t) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

Example 4.24 Consider a unity feedback system with a closed transfer function

$$\frac{C(s)}{R(s)} = \frac{Ks+b}{s^2+as+b}$$

Determine the open-loop transfer function $G(s)$. Show that the steady-state error with unit-ramp input is given by $\frac{a-K}{b}$.

Solution: For the given unity feedback control system, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{Ks+b}{s^2+as+b}$$

Cross multiplying the above equation,

$$(s^2 + as + b)G(s) = (Ks + b) + (Ks + b)G(s)$$

i.e.

$$G(s)[(s^2 + as + b) - (Ks + b)] = (Ks + b)$$

$\therefore$

$$G(s) = \frac{Ks+b}{s[s+(a-K)]}$$

The velocity error constant

$$K_v = \underset{s\to 0}{\text{Lt}}\ sG(s)$$

$$= \underset{s\to 0}{\text{Lt}}\ s \times \frac{Ks+b}{s[s+(a-K)]} = \frac{b}{a-K}$$

Therefore, the steady-state error for a unit-ramp input is

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{\frac{b}{a-K}} = \frac{a-K}{b}$$

Example 4.25 A unity feedback system has the forward path transfer function

$$G(s) = \frac{K_1(2s+1)}{s(5s+1)(s+1)^2}$$

The input $r(t) = 1 + 6t$ is applied to the system. Determine the minimum value of K_1 if the steady-state error is to be less than 0.1.

Solution: The given system is a type-1 system. Therefore,

$$K_p = \infty$$

$$K_v = \underset{s\to 0}{\text{Lt}}\ s\frac{K_1(2s+1)}{s(5s+1)(1+s)^2} = K_1$$

and

$$K_a = 0$$

Since the input is $r(t) = 1 + 6t$, applying the superposition theorem, the steady-state error for the input

$r(t) = 1 + 6t = 1 \times$ (steady-state error for a unit-step input)
$+ 6 \times$ (steady-state error for a unit-ramp input)

$$= 1 \times \left(\frac{1}{1+K_p}\right) + 6 \times \left(\frac{1}{K_v}\right) = \frac{1}{1+\infty} + \frac{6}{K_1} = \frac{6}{K_1}$$

Given that $e_{ss} < 0.1$, for minimum value of K_1

$$e_{ss}(t) = 0.1 = \frac{6}{K_1}$$

or

$$K_1 = \frac{6}{0.1} = 60$$

So the minimum value of K_1 is 60.

Example 4.26 The open-loop transfer function of a servo system with unity feedback is

$$G(s) = \frac{10}{s(0.1s+1)}$$

Evaluate the static error coefficients (K_p K_v, K_a) for the system. Obtain the steady-state error of the system when subjected to an input given by the polynomial

$$r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$$

Also evaluate the dynamic error using the dynamic error coefficients.

Solution: (a) The position error constant

$$K_p = \underset{s \to 0}{\text{Lt}}\, G(s)$$

$$= \underset{s \to 0}{\text{Lt}}\, \frac{10}{s(0.1s+1)} = \infty$$

Therefore, the steady-state error for a unit-step input is

$$e_{ss}(t) = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

The velocity error constant

$$K_v = \underset{s \to 0}{\text{Lt}}\, sG(s)$$

$$= \underset{s \to 0}{\text{Lt}}\, s \times \frac{10}{s(0.1s+1)} = 10$$

Therefore, the steady-state error for a unit-ramp input is

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{10} = 0.1$$

The acceleration error constant

$$K_a = \underset{s\to 0}{\text{Lt}}\, s^2 G(s)$$

$$= \underset{s\to 0}{\text{Lt}}\, s^2 \times \frac{10}{s(0.1s+1)} = 0$$

Therefore, the steady-state error for a unit-acceleration input is

$$e_{ss}(t) = \frac{1}{K_a} = \frac{1}{0} = \infty$$

For a linear system, we can apply the principle of superposition. Therefore, the steady-state error for an input

$$r(t) = a_0 + a_1 t + a_2 \frac{t^2}{2}$$

= a_0 × (steady-state error for a unit-step input) +
a_1 × (steady-state error for a unit-ramp input) +
a_2 × (steady-state error for a unit-parabolic input)

$$= a_0 \times (0) + a_1 \times (0.1) + a_2 \times (\infty) = 0 + 0.1a_1 + \infty = \infty$$

(b) For the given unity feedback system,

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)} = \frac{1}{1+\dfrac{10}{s(0.1s+1)}} = \frac{0.1s^2+s}{0.1s^2+s+10}$$

For expanding it into an infinite series in ascending powers of s, write the numerator and denominator polynomials in ascending powers of s, i.e.

$$\frac{E(s)}{R(s)} = \frac{s+0.1s^2}{10+s+0.1s^2} = \frac{s}{10} - \frac{s^3}{1000} + \frac{s^4}{10000}$$

$$\begin{array}{r|l}
 & \frac{s}{10} - \frac{s^3}{1000} + \frac{s^4}{10000} \cdots \\
\hline
10 + s + 0.1s^2 & s + 0.1s^2 \\
 & s + 0.1s^2 + 0.01s^3 \\
\cline{2-2}
 & -0.01s^3 \\
 & -0.01s^3 - 0.001s^4 - 0.0001s^5 \\
\cline{2-2}
 & 0.001s^4 + 0.0001s^5
\end{array}$$

Equating it with

$$\frac{E(s)}{R(s)} = C_0 + C_1 s + \frac{C_2}{2!}s^2 + \frac{C_3}{3!}s^3 + \cdots$$

The dynamic error coefficients are as follows:

$$C_0 = 0,\ C_1 = \frac{1}{10} = 0.1,\ C_2 = 0$$

Therefore, the dynamic error is

$$e(t) = C_0 r(t) + C_1 \frac{dr(t)}{dt} + \frac{C_2}{2!}\frac{d^2 r(t)}{dt^2} + \frac{C_3}{3!}\frac{d^3 r(t)}{dt^3} + \cdots$$

$$= \frac{1}{\infty}\left(a_0 + a_1 t + \frac{a_2 t^2}{2}\right) + \frac{1}{10}\left[\frac{d}{dt}\left(a_0 + a_1 t + \frac{a_2 t^2}{2}\right)\right] + \frac{1}{\infty}\left[\frac{d^2}{dt^2}\left(a_0 + a_1 t + \frac{a_2 t^2}{2}\right)\right] \times \frac{1}{2}$$

$$= 0 + 0.1(a_1 + a_2 t) + 0$$

$$= 0.1(a_1 + a_2 t)$$

Therefore, the steady-state error

$$e_{ss}(t) = \underset{t\to\infty}{\text{Lt}}\ e(t) = \underset{t\to\infty}{\text{Lt}}\ 0.1(a_1 + a_2 t) = \infty$$

Example 4.27 A unity feedback system has

$$G(s) = \frac{K}{s(s+1)(0.1s+1)} \quad \text{and } r(t) = 10t$$

(a) If $K = 2\ s^{-1}$, determine $e_{ss}(t)$.

(b) Find the minimum value of K for $e_{ss}(t) < 0.1$, for a unit-ramp input.

Solution:

(a) Input $r(t) = 10t$

Therefore, $R(s) = \dfrac{10}{s^2}$

The steady-state error for a unity feedback system is given by

$$e_{ss}(t) = \underset{s\to 0}{\text{Lt}}\ sE(s) = \underset{s\to 0}{\text{Lt}}\ s\frac{R(s)}{1+G(s)}$$

$$= \underset{s\to 0}{\text{Lt}}\ \frac{s\left(\dfrac{10}{s^2}\right)}{1 + \dfrac{K}{s(s+1)(0.1s+1)}}$$

$$= \underset{s\to 0}{\text{Lt}}\ \frac{s\left(\dfrac{10}{s^2}\right)s(s+1)(0.1s+1)}{s(s+1)(0.1s+1)+K}$$

$$\therefore \qquad K_v = \frac{R}{e_{ss}} = \frac{3.14 \times 180}{0.2 \times 3.14} = 900$$

But
$$K_v = \underset{s \to 0}{\text{Lt}}\ sG(s) = \underset{s \to 0}{\text{Lt}}\ s \frac{10K}{s(1+0.1s)} = 10K$$

$$\therefore \qquad 10K = 900$$

or
$$K = 90$$

The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{900}{s(1+0.1s)}}{1+\dfrac{900}{s(1+0.1s)}} = \frac{9000}{s^2 + 10s + 9000}$$

Comparing this with the standard form of the transfer function of a second-order system,

i.e. with
$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}, \quad \text{we get } \omega_n^2 = 9000$$

Therefore, the natural frequency

$$\omega_n = \sqrt{9000} = 94.87 \text{ rad/s}$$

$$2\xi\omega_n = 10$$

Therefore, the damping ratio

$$\xi = \frac{10}{2\omega_n} = \frac{10}{2 \times 94.87} = 0.053$$

Example 4.30 A control system is designed to keep the antenna of a tracking radar pointed at a flying target. The system must be able to follow a target travelling in a straight line with a speed of 200 m/s with maximum permissible error of 0.01 degree. The shortest distance from the antenna to the target is 250 m. Find the value of error constant K_v in order to satisfy the requirements.

Figure 4.28 shows the arrangement.

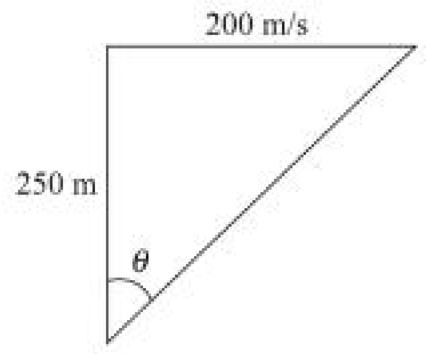

Figure 4.28 Example 4.30.

Solution: Speed = 200m/s

$$\text{Angular velocity} = \tan^{-1}\frac{200}{250} = 38.66 \text{ deg/s} = 0.675 \text{ rad/s}$$

The steady-state error

$$e_{ss} = \frac{\text{Angular velocity}}{K_v}$$

$$\therefore \quad K_v = \frac{\text{Velocity}}{\text{Error}} = \frac{38.66}{0.01} = 3866$$

or

$$\text{the steady-state error} = 0.01° = 0.01 \times \frac{\pi}{180} \text{ rad}$$

$$\therefore \quad K_v = \frac{0.675}{0.01 \times \dfrac{\pi}{180}} = \frac{0.675 \times 180}{0.01 \times \pi} = 3867$$

4.9 RESPONSE WITH P, PI, PD AND PID CONTROLLERS

4.9.1 Proportional Control

In proportional control, the actuating signal $E_a(s)$ is proportional to the error signal $E(s)$. Hence the name proportional control system. The block diagram of Figure 4.29 shows the proportional control action.

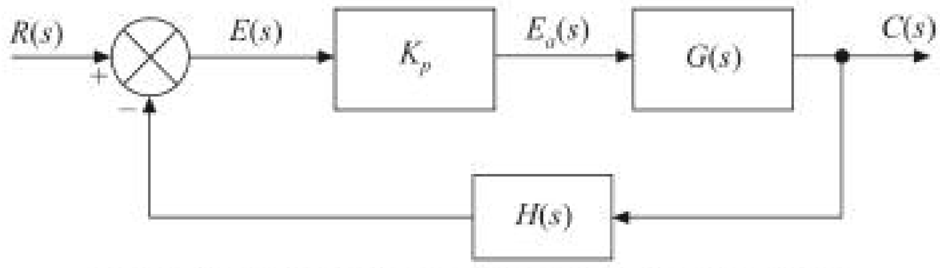

Figure 4.29 Block diagram of proportional control system.

For quick response, it is necessary that the control system should be underdamped. An underdamped control system has exponentially decaying oscillations in the output time response during the transient period. The sluggish (slow moving) overdamped response of a control system can be made faster by increasing the forward path gain of the system. An increase in the forward path gain of system results in a reduction of the steady-state error but peak overshoot gets increased. For a satisfactory performance of a control system, a convenient adjustment has to be made between the maximum overshoot and the steady-state error. Without scarifying the steady-state accuracy, the maximum overshoot can be reduced to some extent by modifying the actual signal.

$$\frac{E_a(s)}{E(s)} = K_p$$

where K_p is known as proportional gain.

4.9.2 Derivative Control

In derivative control, the actuating signal consists of proportional error signal and derivative of the error signal, i.e.

$$e_a(t) = e(t) + T_d \frac{d}{dt} e(t) \tag{4.27}$$

Taking the Laplace transform on both sides of Eq. (4.27),

$$E_a(s) = E(s) + T_d sE(s)$$

i.e.

$$E_a(s) = (1 + sT_d)E(s)$$

Figure 4.30 shows a block diagram of a second-order unity feedback control system using derivative control.

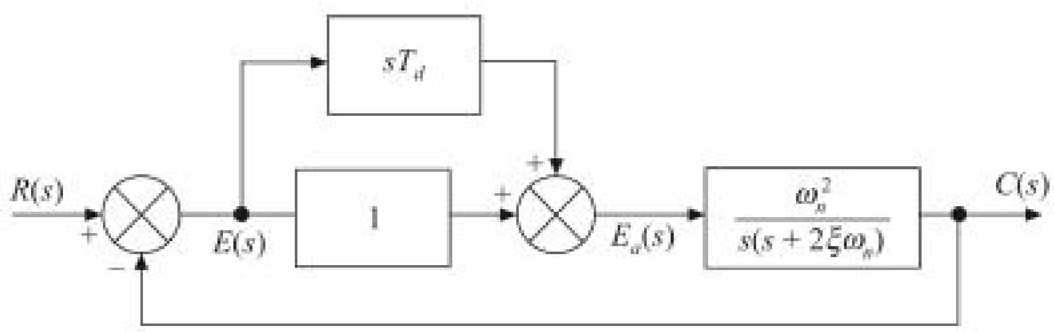

Figure 4.30 Block diagram of a control system with derivative control.

The overall transfer function of the system of Figure 4.30 is given by

$$\frac{C(s)}{R(s)} = \frac{(1+sT_d)\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+(1+sT_d)\times\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}}$$

$$= \frac{\omega_n^2(1+sT_d)}{s^2+2\xi\omega_n s+\omega_n^2+sT_d\omega_n^2}$$

$$= \frac{\omega_n^2(1+sT_d)}{s^2+(2\xi\omega_n+\omega_n^2 T_d)s+\omega_n^2}$$

So, the characteristic equation is

$$s^2 + (2\xi\omega_n + \omega_n^2 T_d)s + \omega_n^2 = 0 \tag{4.28}$$

If ξ' is the effective damping ratio of the second-order system with derivative control, its characteristic equation should be

$$s^2 + 2\xi'\omega_n s + \omega_n^2 = 0 \tag{4.29}$$

Comparing Eqs. (4.28) and (4.29),

$$2\xi'\omega_n = 2\xi\omega_n + \omega_n^2 T_d$$

i.e.

$$\xi' = \frac{2\xi\omega_n + \omega_n^2 T_d}{2\omega_n}$$

i.e.

$$\xi' = \xi + \frac{\omega_n T_d}{2} \tag{4.30}$$

Equation (4.30) shows that the damping ratio is increased using derivative control, but the maximum overshoot is reduced. Now the overall transfer function of the system of Figure 4.30 can be written as

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2 T_d \left(s + \frac{1}{T_d}\right)}{s^2 + 2\xi'\omega_n s + \omega_n^2}$$

The error function is given by

$$\frac{E(s)}{R(s)} = \frac{1}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + \frac{\omega_n^2(1 + sT_d)}{s(s + 2\xi\omega_n)}}$$

$$= \frac{s(s + 2\xi\omega_n)}{s(s + 2\xi\omega_n) + \omega_n^2(1 + sT_d)}$$

i.e.

$$\frac{E(s)}{R(s)} = \frac{s(s + 2\xi\omega_n)}{s^2 + (2\xi\omega_n + \omega_n^2 T_d)s + \omega_n^2}$$

For a ramp input,

$$r(t) = t$$

$\therefore$

$$R(s) = \frac{1}{s^2}$$

Therefore, the error signal

$$E(s) = \frac{1}{s^2} \cdot \frac{s(s + 2\xi\omega_n)}{s^2 + (2\xi\omega_n + \omega_n^2 T_d)s + \omega_n^2}$$

The steady-state error for a unit-ramp input is given by

$$e_{ss}(t) = \underset{s \to 0}{\text{Lt}}\ sE(s)$$

The damping ratio with derivative control is

$$\xi' = \xi + \frac{\omega_n T_d}{2}$$

Since the damping ratio with derivative control is to be 0.75 and $\omega_n = 2$, we have

$$0.75 = 0.25 + \frac{2 \times T_d}{2}$$

$\therefore$ $$T_d = 0.5$$

Therefore, the block diagram with derivative control is as shown in Figure 4.34.

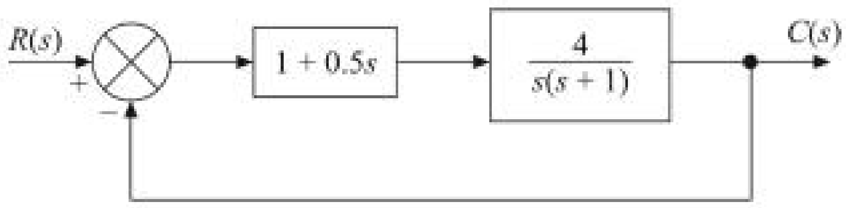

Figure 4.34 Example 4.31: Block diagram with derivative control.

Without derivative control, the rise time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \tan^{-1} \dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n \sqrt{1-\xi^2}}$$

$$= \frac{\pi - \tan^{-1} \dfrac{\sqrt{1-0.25^2}}{0.25}}{2 \times \sqrt{1-0.25^2}} = \frac{1.823}{1.936} = 0.942 \text{ s}$$

The peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$= \frac{\pi}{2\sqrt{1-0.25^2}} = 1.62 \text{ s}$$

The peak overshoot is given by

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$= e^{-\pi \times 0.25/\sqrt{1-0.25^2}}$$

$$= 0.4443$$

The overall transfer function with derivative control is

$$\frac{C(s)}{R(s)} = \frac{\dfrac{4(1+0.5s)}{s(s+1)}}{1+\dfrac{4(1+0.5s)}{s(s+1)}} = \frac{4(1+0.5s)}{s^2+3s+4}$$

$$= \frac{4+2s}{s^2+3s+4}$$

For a unit-step input, $r(t) = 1$

$$\therefore \qquad R(s) = \frac{1}{s}$$

$$C(s) = \frac{4+2s}{s(s^2+3s+4)}$$

$$= \frac{A}{s} + \frac{Bs+C}{s^2+3s+4}$$

$$= \frac{1}{s} - \frac{(s+1)}{s^2+3s+4} = \frac{1}{s} - \frac{s+1}{(s+1.5)^2+1.75}$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1.5)^2+1.32^2} = \frac{1}{s} - \frac{(s+1.5)-0.5}{(s+1.5)^2+1.32^2}$$

$$= \frac{1}{s} - \frac{(s+1.5)}{(s+1.5)^2+1.32^2} + \frac{0.5}{1.32}\frac{1.32}{(s+1.5)^2+1.32^2}$$

$$c(t) = 1 - e^{-1.5t}\cos 1.32t + 0.378e^{-1.5t}\sin 1.32t$$

$$= 1 - e^{-1.5t}(\cos 1.32t - 0.378 \sin 1.32t)$$

At $\qquad t = t_r,$

$$c(t) = 1$$

i.e. $\qquad c(t_r) = 1$

$$\therefore \qquad 1 = 1 - e^{-1.5t_r}(\cos 1.32t_r - 0.378 \sin 1.32t_r)$$

$$\therefore \qquad \cos 1.32t_r = 0.378 \sin 1.32t_r$$

$$\tan 1.32t_r = 1/0.378 = 2.645$$

$$\therefore \qquad 1.32t_r = \tan^{-1} 2.645 = 69.28° = 1.209 \text{ rad}$$

$$\therefore \qquad t_r = 1.209/1.32 = 0.916\text{s}$$

For peak time, differentiating the output equation with respect to t and equating to zero

$$\left.\frac{d}{dt}c(t)\right|_{t=t_p} = \left.\frac{d}{dt}[1 - e^{-1.5t}(\cos 1.32t - 0.378 \sin 1.32t)]\right|_{t=t_p} = 0$$

i.e. $$-e^{-1.5t}\,[-\sin 1.32t\,(1.32) - 0.378 \times 1.32 \cos 1.32t]$$

$$-e^{-1.5t}\,(-1.5)\,[\cos 1.32t - 0.378 \sin 1.32t]\big|_{t=t_p} = 0$$

i.e. $$-1.32 \sin 1.32t_p - 0.498 \cos 1.32t_p + 0.567 \sin 1.32t_p - 1.5 \cos 1.32t_p = 0$$

i.e. $$-0.753 \sin 1.32t_p - 1.998 \cos 1.32t_p = 0$$

$$\tan 1.32t_p = -1.998/0.753 = -2.653$$

$\therefore$ $$t_p = \frac{\tan^{-1}(-2.63)}{1.32} = \frac{\pi - \tan^{-1} 2.63}{1.32} = \frac{\pi - 1.207}{1.32} = 1.46 \text{ s}$$

Peak output occurs at $t = t_p$

$$c(t_p) = 1 - e^{-1.5t_p}\,(\cos 1.32t_p - 0.378 \sin 1.32t_p)$$

$$= 1 - e^{-1.5\times 1.46}\,(\cos 1.32 \times 1.46 - 0.378 \sin 1.32 \times 1.46)$$

$$= 1 - e^{-2.19}\,(\cos 1.927 - 0.378 \sin 1.927)$$

$$= 1 - 0.119\,(\cos 110.4° - 0.378 \sin 110.4°)$$

$$= 1 - 0.119\,(-0.348 - 0.354)$$

$$= 1.08$$

Therefore, the peak overshoot

$$M_p = c(t_p) - c(\infty)$$

$$= 1.08 - 1 = 0.08$$

$$\%M_p = 8\%$$

Example 4.32 In Example 4.31, it is desired the damping ratio be 0.75. Determine the derivative rate feedback K_t. Also determine the peak time, rise time, steady-state error and maximum overshoot with and without feedback for unit-ramp input.

Solution: The characteristic equation is

$$s^2 + s + 4 = 0$$

$\therefore$ $$\xi = 0.25$$

and $$\omega_n = 2$$

Let ξ' be the damping ratio with derivative feedback. Therefore,

$$\xi' = \xi + \frac{\omega_n K_t}{2}$$

Putting values in the above equation, we get

$$0.75 = 0.25 + \frac{2K_t}{2}$$

$\therefore$ $$K_t = 0.5$$

The steady-state error without feedback is

$$e_{ss} = \frac{2\xi}{\omega_n} = \frac{2 \times 0.25}{2} = 0.25$$

Already we have determined t_r, t_p and M_p to be

$$t_r = 0.942 \text{ s}, \ t_p = 1.62 \text{ s}, \text{ and } M_p = 0.444$$

The overall transfer function using derivative feedback control is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + (2\xi\omega_n + \omega_n^2 K_t)s + \omega_n^2}$$

Given that $\xi = 0.25$ and $\omega_n = 2$ rad/s, $K_t = 0.5$

$$\therefore \qquad \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 3s + \omega_n^2} = \frac{4}{s^2 + 3s + 4}$$

The characteristic equation is

$$s^2 + 3s + 4 = 0$$

i.e. $\xi = 0.75$

and $\omega_n = 2$

The rise time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi - \tan^{-1}\dfrac{\sqrt{1-0.75^2}}{0.75}}{2 \times \sqrt{1-0.75^2}}$$

$$= \frac{2.419}{1.322} = 1.829 \text{ s}$$

The peak time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{2\sqrt{1-0.75^2}} = 3.59 \text{ s}$$

The maximum overshoot

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}}$$

$$= e^{-\pi \times 0.75/\sqrt{1-0.75^2}}$$

$$= 0.074 = 7.4\%$$

The steady-state error

$$e_{ss} = \frac{2\xi}{\omega_n} + K_t$$

$$= \frac{2 \times 0.75}{2} + 0.5 = 1.25$$

4.9.4 Integral Control

For integral control action, the actuating signal consists of proportional error signal added with integral error signal, i.e.

$$e_a(t) = e(t) + K_i \int e(t)\, dt \qquad (4.34)$$

where K_i is a constant.

Taking the Laplace transform on both sides of Eq. (4.34),

$$E_a(s) = E(s) + K_i \frac{E(s)}{s} = E(s)\left(1 + \frac{K_i}{s}\right) \qquad (4.35)$$

Figure 4.35 shows the block diagram of a system with integral control action.

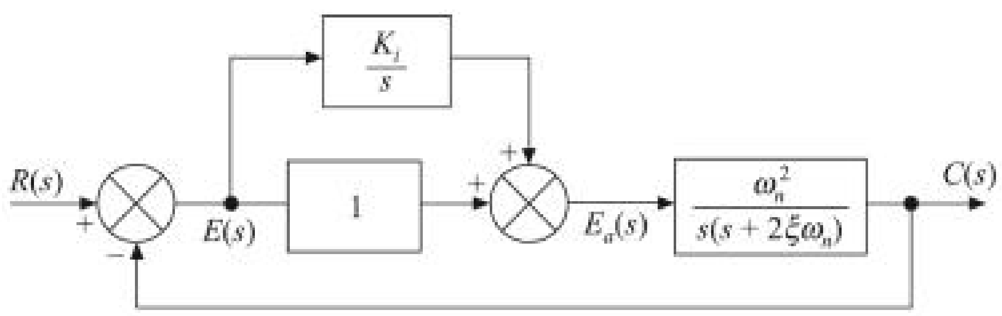

Figure 4.35 Block diagram of a control system with integral control.

From the block diagram, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{\left(1 + \frac{K_i}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s}\right)}{1 + \left(1 + \frac{K_i}{s}\right)\left(\frac{\omega_n^2}{s^2 + 2\xi\omega_n s}\right)}$$

$$= \frac{(s + K_i)\,\omega_n^2}{s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2}$$

The characteristic equation $s^3 + 2\xi\omega_n s^2 + \omega_n^2 s + K_i\omega_n^2 = 0$ is of third-order. So we see that the second-order system has become a third-order system due to the inclusion of the integral control.

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(b) *Without error rate control*

When $K_e = 0$,

$$G(s) = \frac{10}{s(s+2)}$$

Therefore, the closed-loop transfer function is

$$\frac{\theta_C(s)}{\theta_R(s)} = \frac{\dfrac{10}{s(s+2)}}{1 + \dfrac{10}{s(s+2)}} = \frac{10}{s^2 + 2s + 10}$$

Comparing this transfer function with the standard form of the transfer function of a second-order system $\dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$, we get

$$\omega_n^2 = 10$$

$\therefore$
$$\omega_n = \sqrt{10} = 3.16 \text{ rad/s}$$

$$2\xi\omega_n = 2$$

$\therefore$
$$\xi = \frac{2}{2\omega_n} = \frac{2}{2 \times 3.16} = 0.32$$

The settling time

$$t_s = \frac{4}{\xi\omega_n} = \frac{4}{0.32 \times 3.16} = 4 \text{ s}$$

The peak overshoot

$$M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} = e^{-\pi \times 0.32/\sqrt{1-0.32^2}} = 0.351$$

The steady-state error

$$e_{ss}(t) = \frac{1}{K_v} = \frac{1}{\underset{s \to 0}{\text{Lt}}\, sG(s)} = \frac{1}{\underset{s \to 0}{\text{Lt}}\, s\dfrac{10}{s(s+2)}} = \frac{1}{5} = 0.2 \text{ rad}$$

SHORT QUESTIONS AND ANSWERS

1. What do you mean by time response of a control system?

A. The time response of a control system is the output of the system as a function of time. It is given by the inverse Laplace transform of the product of input and system transfer functions. The time response of a control system is the sum of the transient response and the steady-state response.

2. What do you mean by transient response? On what does it depend?

A. The transient response of a system is that part of the time response which tends to zero as t tends to infinity. It is the response of the system when the input changes from one state to the other. The nature of the transient response is dependent only on the system poles and not on the type of input. The transient response of a system is also called the dynamic response.

3. What do you mean by steady-state response? On what does it depend?

A. The steady-state response of a system is that part of the time response which remains constant as $t \to \infty$. It is dependent on the system poles as well as on the type of the input.

4. What is the importance of test signals?

A. The importance of test signals is: the test signals can be easily generated in the laboratories and the characteristics of test signals resemble the characteristics of actual signals. The test signals are used to pre-determine the performance of the system. If the response of a system is satisfactory for a test signal, then the system will be suitable for practical applications.

5. Name the standard test signals used in control systems.

A. The standard test signals used in control systems are: impulse, step, ramp, parabolic and sinusoidal.

6. Why are test signals needed?

A. Usually the input signals to a control system are not known fully ahead of time. They may be random in nature. But to test the performance of a system while designing, some inputs are required to be given. They are called test signals.

7. Define the test signals: (a) step (b) ramp (c) parabolic and (d) impulse.

A. A unit-step signal is one which remains at zero level for $t < 0$ and changes suddenly to a level of unity at $t = 0$ and remains at 1 for $t > 0$, i.e. $u(t) = 0$ for $t < 0$ and $u(t) = 1$, for $t > 0$.

A unit-ramp signal is one which remains at zero level for $t < 0$ and increases linearly with time for $t > 0$, i.e. $r(t) = 0$, for $t < 0$ and $r(t) = t$ for $t > 0$.

A unit-parabolic signal is one which remains at zero level for $t < 0$ and increases as per the equation $t^2/2$ for $t > 0$, i.e. $r(t) = 0$, for $t < 0$, and $r(t) = t^2/2$ for $t > 0$.

A unit-impulse signal is defined as a signal which has zero value everywhere except at $t = 0$, where its magnitude is infinity. $\delta(t) = 0$, for $t \neq 0$; $\delta(t) = \infty$, at $t = 0$.

8. What do you mean by weighing function?

A. The system's impulse response is referred to as weighing function of the system.

9. Define the impulse response of a system?

A. The impulse response of a system is the response or the output of a system for a unit-impulse input. The impulse response of a system is also defined as the inverse Laplace transform of its transfer function. It is also referred to as the weighting function of the system.

10. How are higher-order systems approximated to second-order systems?

A. Higher-order systems usually have a pair of complex conjugate poles with damping less than one which dominate over other poles. So they can be represented by second-order systems considering only the dominant poles and neglecting all other poles.

11. Define the terms:

(a) Pole

(b) Zero

A. The pole of a function $F(s)$ is the value of s for which the function $F(s)$ becomes infinite. The zero of a function $F(s)$ is the value of s for which the function $F(s)$ becomes zero.

12. What do you mean by the term 'order' of a control system?

A. The term 'order' of a system indicates the order of the differential equation used to describe the system. It is also given by the highest power of s present in the denominator of the closed-loop transfer function. The highest power of s in the denominator gives the number of poles of the system and so the order of the system is also given by the number of poles of the transfer function.

13. What do you mean by time constant of a system? What does it indicate?

A. The time constant of a system is defined as the time taken by the output to reach 100% of the input step if the initial slope of the output is maintained constant. It is indicative of how fast the system tends to reach the final value. A large time constant corresponds to a sluggish system and a small time constant corresponds to fast response.

14. How is the speed of response defined?

A. The speed of response can be quantitatively defined as the time required for the output to become a particular percentage of its final value.

15. Define the characteristic polynomial.

A. The denominator polynomial of the closed-loop transfer function is called the characteristic polynomial.

16. Define the characteristic equation. Why that name?

A. The denominator of the closed-loop transfer function equated to zero is called the characteristic equation. The roots of the characteristic equation characterize the behaviour of the system. Hence that name.

17. How are the poles of a closed-loop system and roots of the characteristic equation related?

A. The roots of the characteristic equation are the same as the poles of the closed-loop transfer function.

18. What does the time constant of a system indicate?

A. The time constant of a system is indicative of how fast the system tends to reach the final value. A large time constant corresponds to a sluggish system and a small time constant corresponds to fast response.

19. Define the term 'damping ratio' of a system.

A. The damping factor or damping ratio ξ of a system is defined as the ratio of actual damping to critical damping.

20. How are control systems classified depending on the value of damping?

A. Depending on the value of damping, control systems may be classified as follows:

(a) Undamped systems, $\xi = 0$ (b) Underdamped systems, $\xi < 1$

(c) Criticallydamped systems, $\xi = 1$ (d) Overdamped systems, $\xi > 1$

21. What is the location of poles for different types of damped systems?

A. (a) For undamped systems, $\xi = 0$. The poles are on the imaginary axis and are conjugate of each other.

(b) For underdamped systems, $0 < \xi < 1$. The poles are in the left half of the *s*-plane and are the complex conjugate of each other.

(c) For critically-damped systems, $\xi = 1$. Both the poles are real, negative and equal.

(d) For overdamped systems, $\xi > 1$. The poles are real, negative and unequal.

22. What is the nature of response of a second-order system with different types of damping?

A. The nature of response of a second-order system with different types of damping is as follows:

(a) For undamped systems, the response is purely oscillatory.
(b) For underdamped systems, the response is damped oscillatory.
(c) For critically-damped systems, the response is exponentially rising.
(d) For overdamped systems, the response is exponentially rising but the rise time will be very large.

23. For quick response, what type of damping is preferred?

A. For quick response, the control system should be underdamped. So control systems are normally designed with $\xi < 1$, i.e. mostly they are underdamped systems.

24. What do you mean by undamped natural frequency?

A. The undamped natural frequency ω_n is the frequency at which the system will oscillate in the absence of any damping.

25. What do you mean by damped frequency of oscillation?

A. The damped frequency of oscillation, $\omega_d = \omega_n\sqrt{1-\xi^2}$ is the frequency of damped oscillations of an underdamped system.

26. List the time-domain specifications of a second-order underdamped system excited by a unit-step input.

A. The time-domain specifications of a second-order underdamped system excited by a step input are (a) delay time, (b) rise time, (c) peak time, (d) maximum overshoot, (e) settling time and (f) steady-state error.

42. The acceleration error constant K_a is indicative of the error in output for a __________ input.

43. Control systems are classified in accordance with the number of integrations in the open-loop transfer function as __________, __________, __________ systems etc.

44. A system with no open-loop poles at the origin of the s-plane is called a __________ system.

45. A system with one open-loop pole at the origin of the s-plane is called a __________ system.

46. A system with two open-loop poles at the origin of the s-plane is called a __________ system.

47. The number of open-loop poles at the origin of the s-plane indicates the __________ of the system.

48. A type-0 system has __________ position error, __________ velocity error and __________ acceleration error.

49. A type-1 system has __________ position error, __________ velocity error and __________ acceleration error.

50. A type-2 system has __________ position error, __________ velocity error and __________ acceleration error.

51. The __________ describe the ability of a system to reduce or eliminate steady-state errors.

52. As __________ of system becomes higher, progressively more steady-state errors are eliminated but it is more difficult to __________ the system and the __________ errors for such systems tend to be larger.

53. The highest power of s present in the denominator of the closed-loop transfer function indicates the __________ of the control system.

54. The time-domain specifications of a control system are __________, __________, __________, __________, __________, and __________.

55. The relation between the static and dynamic error coefficients is C_0 = __________, C_1 = __________, and C_2 = __________.

56. In proportional control, the actuating signal is __________ to the error signal.

57. It is necessary that the control system must be __________ for quick response.

58. The sluggish overdamped system can be made faster by increasing the __________ of the system.

59. Due to the increase in the forward path gain, the __________ gets reduced but the __________ increases.

60. The actuating signal for derivative control action consists of __________ error signal added with __________ error signal.

61. Derivative control results in an __________ of damping ratio but the __________ is reduced.

62. Derivative control does not affect __________.

63. Derivative control results in the addition of a __________ to the transfer function.

64. Derivative control __________ the value of ω_n.

65. The introduction of zero in derivative control results in __________ of rise time t_r.

66. In integral control, the actuating signal consists of __________ error signal added with __________ of the error signal.

67. The derivative feedback control is also called the __________ feedback control or __________ feedback control.

68. In derivative feedback control, the actuating signal is the difference between the __________ error signal and __________ of the output signal.

69. Derivative feedback control __________ the damping ratio and so the maximum overshoot will be __________.

70. Derivative feedback control __________ the steady-state error of the control system.

71. Integral control __________ the order of a control system.

72. The addition of a pole to the forward path transfer function __________ the order of the system, __________ the overshoot, __________ the stability, __________ the rise time and __________ the bandwidth.

73. The addition of a pole to the closed-loop transfer function __________ the rise time and __________ the overshoot.

74. The addition of a zero to the closed-loop transfer function __________ the rise time and __________ the maximum overshoot.

75. The poles which are very near to the imaginary axis are called __________ poles and the poles which are far away from the imaginary axis are called __________ poles.

76. In design of higher-order systems, the __________ poles of the system are used to control the dynamic performance of the system whereas the __________ poles are used for the purpose of ensuring that the controller transfer function can be realized by physical components.

77. If the system dynamics of a higher-order system can be accurately represented by a pair of complex conjugate dominant poles, then that ξ is called __________ damping ratio.

78. If the significant poles are neglected as they are, the __________ response is unchanged, but the __________ response gets affected.

79. To maintain the same steady-state response, the transfer function has to be written in __________ form.

80. A pole may be regarded as insignificant, if the magnitude of the real part of the pole is at least __________ times that of a dominant pole or a pair of complex dominant poles.

Answers to Fill in the Blanks

1. parameters **2.** not known **3.** a sudden shock, a sudden change, a constant velocity and a constant acceleration **4.** system poles, input **5.** system poles, input **6.** unit-impulse, unit-step, unit-ramp, unit-parabolic, sinusoidal **7.** impulse response **8.** impulse **9.** weighting function **10.** step **11.** ramp **12.** parabolic **13.** transient response, steady-state response **14.** dynamic response **15.** transient **16.** how fast **17.** sluggish, fast **18.** damping ratio **19.** undamped natural frequency **20.** characteristic polynomial **21.** characteristic equation **22.** poles **23.** characterize **24.** damped **25.** < **26.** undamped **27.** underdamped **28.** critically-damped **29.** overdamped **30.** less **31.** delay time **32.** rise time **33.** peak time **34.** peak overshoot **35.** settling time **36.** $\omega_n t_s$ **37.** steady-state **38.** nature of inputs, type of system, the nonlinearities **39.** unit-step, unit-ramp, unit-parabolic **40.** unit-step **41.** unit-velocity **42.** unit-acceleration **43.** type-0, type-1, type-2 **44.** type-0 **45.** type-1 **46.** type-2 **47.** type number **48.** finite, infinite, infinite **49.** zero, finite, infinite **50.** zero, zero, finite **51.** steady-state error constants **52.** type, stabilize, dynamic **53.** order **54.** delay time, peak time, rise time, maximum overshoot, settling time, steady-state error **55.** $C_0 = 1/(1 + K_p)$, $C_1 = 1/K_v$, $C_2 = 1/K_a$ **56.** proportional **57.** underdamped **58.** forward path gain **59.** steady-state error, maximum overshoot **60.** proportional, derivative **61.** increase, overshoot **62.** the steady-state error **63.** zero **64.** does not change **65.** reduction **66.** proportional, integral **67.** rate, tachometer **68.** proportional, derivative **69.** increases, reduced **70.** increases **71.** increases **72.** increases, increases, reduces, increases, reduces **73.** increases, reduces **74.** decreases, increases **75.** dominant, insignificant **76.** dominant, insignificant **77.** relative **78.** transient, steady-state **79.** time constant **80.** 5 to 10.

OBJECTIVE TYPE QUESTIONS

1. Knowledge of the transfer function of a system is necessary for the calculation of

(a) time constant (b) output for a given input
(c) order of the system (d) none of these

2. Zero initial conditions means that the system is

(a) working with zero stored energy
(b) working with zero reference signal
(c) at rest and no energy is stored in any of its components
(d) none of these

3. The transfer function is defined for

(a) linear time-invariant systems (b) linear time-varying systems
(c) nonlinear systems (d) none of these

4. The error signal in a control system is

(a) the difference between the measured value and the set value
(b) the sum of measured value and set value
(c) the ratio of measured value to set value
(d) none of these

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4.8 A closed-loop control system is represented by the differential equation

$$\frac{d^2c}{dt^2} + 6\frac{dc}{dt} = 20e$$

where $e = r - c$ is the error signal. Determine the undamped natural frequency, damping ratio, and percentage maximum overshoot for a unit-step input.

4.9 A unity feedback control system has an open-loop transfer function

$$G(s) = \frac{4}{s(s+2)}$$

Determine the rise time, peak time, peak overshoot and the settling time when a step displacement of 15° is given to the system.

4.10 Determine the values of K and T of the closed-loop system shown in Figure P4.3 so that the maximum overshoot in unit-step response is 20% and the peak time is 1.5 s. Assume that $J = 1$ kg-m^2.

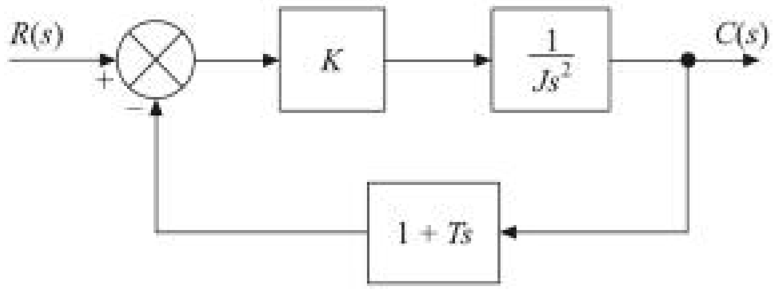

Figure P4.3

4.11 A unity feedback system is characterized by the open-loop transfer function

$$G(s) = \frac{1}{s(0.4s+1)(0.15s+1)}$$

Determine the steady-state errors for unit-step, unit-ramp and unit-acceleration inputs.

4.12 The open-loop transfer function of a unity feedback control system is given by

$$G(s) = \frac{20}{s(s+2)(s+5)}$$

Find the steady-state error when it is subjected to the input, $r(t) = 5 + 3t + 2t^2$.

4.13 A unity feedback system has

$$G(s) = \frac{10}{s(s+2)}$$

Find the generalized error coefficients and the steady-state error.

4.14 A unity feedback system has the forward path transfer function

$$G(s) = \frac{10}{s(s+2)}$$

Find the steady-state error and the generalized error coefficients for $r(t) = t$.

4.15 For a unity feedback system having an open-loop transfer function

$$G(s) = \frac{K(s+2)(s+3)}{s^2(s^2+8s+15)}$$

Determine (a) type of system, (b) error constants K_p K_v and K_a and (c) steady-state error for unit step, unit-ramp and unit-parabolic inputs.

4.16 Find the position, velocity and acceleration error constants for the following unity feedback systems having the forward path transfer function $G(s)$ as

(a) $G(s) = \dfrac{25}{(1+0.1s)(1+0.6s)}$ (b) $G(s) = \dfrac{20}{s(1+0.1s)(1+0.5s)}$

(c) $G(s) = \dfrac{50}{s^2(s^2+8s+10)}$ (d) $G(s) = \dfrac{25}{s^3(s^2+5s+4)}$

4.17 The open-loop transfer function of a unity feedback control system is

$$G(s) = \frac{5(s+2)}{(s+1)(s+3)}$$

Using the generalized error series, determine the error signal and the steady-state error of the system when it is excited by

(a) $r(t) = 1$ (b) $r(t) = 2t$

(c) $r(t) = \dfrac{t^2}{2}$ (d) $r(t) = 1 + 2t + \dfrac{t^2}{2}$

Answers to Problems

4.1 (a) Underdamped (b) Overdamped (c) Critically-damped (d) Undamped

4.2 $\omega_d = 8.56$ rad/s, $t_r = 0.293$ s, $t_p = 0.366$ s, $M_p = 0.046$, $t_s = 0.476$ s (for 2%), $t_s = 0.357$ s (for 5%); $C(s)/R(s) = 144/(s^2 + 16.8s + 144)$

4.3 Underdamped, $t_r = 0.972$ s, $t_p = 1.28$ s, $M_p = 0.036$, $t_s = 2.002$ s (for 2% error), $t_s = 1.50$ s (for 5% error).

4.4 $C(t) = 1 - e^{-2.5t}(\cos 1.936t + 1.291 \sin 1.936t)$

4.5 $\omega_d = 3.464$ rad/s, $t_r = 0.604$ s, $t_p = 0.906$ s, $M_p = 0.162$, $t_s = 2$ s (for 2% error), $t_s = 1.5$ s (for 5% error).

4.6 $\dfrac{C(s)}{R(s)} = \dfrac{300}{s^2 + 40s + 300}$, $\omega_n = 17.32$ rad/s, $\xi = 1.156$

4.7 8.102

4.8 ω_n = 4.472 rad/s, ξ = 0.67, % M_p = 19%

4.9 t_r = 1.29 s, t_p = 1.813 s, M_p = 2.445°, % M_p = 16.3%, t_s = 4 s (for 2% error), t_s = 3 s (for 5% error).

4.10 K = 5.514, T = 0.388

4.11 e_{ss} (step) = 0, e_{ss} (velocity) = 1, e_{ss} (acceleration) = 0

4.12 $e_{ss} = \infty$

4.13 $C_0 = \infty$, C_1 = 5, C_2 = 50/3, e_{ss} (t) = ∞

4.14 e_{ss} (t) = ∞, C_0 = 6, $C_1 = \dfrac{36}{2.5}$, $C_2 = \dfrac{216}{1.25}$

4.15 Type-2 system $K_p = \infty$, $K_v = \infty$, $K_a = \dfrac{6K}{15}$, e_{ss} (step) = 0, e_{ss} (ramp) = 0,

e_{ss} (parabolic) = $\dfrac{15}{6K}$

4.16 (a) K_p = 25, K_v = 0, K_a = 0 (b) $K_p = \infty$, K_v = 20, K_a = 0

(c) $K_p = \infty$, $K_v = \infty$, K_a = 5 (d) $K_p = \infty$, $K_v = \infty$, $K_a = \infty$

4.17 $e(t) = \dfrac{3}{13} r(t) + \dfrac{25}{165} \dot{r}(t) - \dfrac{95}{2197} \ddot{r}(t)$

(a) $e(t) = \dfrac{3}{13}$, $e_{ss}(t) = \dfrac{3}{13}$

(b) $e(t) = \dfrac{30}{13} t + \dfrac{50}{169}$, $e_{ss}(t) = \infty$

(c) $e(t) = \dfrac{3}{26} t^2 + \dfrac{25}{169} t - \dfrac{95}{2197}$, $e_{ss}(t) = \infty$

(d) $e(t) = \dfrac{3}{13}\left(1 + 2t + \dfrac{t^2}{2}\right) + \dfrac{25}{169}(2 + t) - \dfrac{95}{2197}$, $e_{ss}(t) = \infty$

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The Routh stability criterion is an algebraic method to determine the stability of linear control systems. Being an algebraic method, it is very simple. There is no need to solve for the roots. Information about stability of control systems can be obtained directly from the coefficients of the characteristic equation. The limitations are as follows:

1. It is valid if and only if the characteristic equation is algebraic with real coefficients. If anyone of the coefficients is complex, or if the equation is not algebraic, such as containing exponential functions or sinusoidal functions of s, the Routh criterion simply cannot be applied.
2. Another limitation of the Routh criterion is that, it is valid only for the determination of the roots of the characteristic equation with respect to the left-half or right-half of the s-plane. The stability boundary is the $j\omega$-axis of the s-plane. The criterion cannot be applied to any other stability boundaries in a complex plane such as the unit circle in the z-plane, which is the stability boundary of discrete-data systems.
3. It gives information only on the absolute stability of a system. Determination of relative stability by repeated application of the Routh criterion by shifting the origin of the s-plane is quite cumbersome. It does not suggest how to improve the relative stability or how to stabilize an unstable system.
4. It can tell how many roots of the characteristic equation are in the right-half of the s-plane, but it cannot give any information about their exact location. Also it cannot tell whether the roots are real or complex.
5. This stability criterion applies to polynomials with only a finite number of terms.

Example 5.3 The characteristic equation of a servo system is given by

$$b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4 = 0$$

Determine the conditions which must be satisfied by the coefficients of the characteristic equation for the system to be stable.

Solution: Form the Routh table as shown below.

s^4	b_0	b_2	b_4
s^3	b_1	b_3	
s^2	$\dfrac{b_1 b_2 - b_0 b_3}{b_1}$	$\dfrac{b_1 b_4 - b_0 \times 0}{b_1} = b_4$	
s^1	$\dfrac{(b_1 b_2 - b_0 b_3) b_3 - b_1^2 b_4}{b_1 b_2 - b_0 b_3}$		
s^0	b_4		

For stability, all the elements in the first column of the Routh array must be positive. So, the conditions that must be satisfied for stability of the given system are as follows:

$$b_1 > 0,\ b_1 b_2 - b_0 b_3 > 0,\ (b_1 b_2 - b_0 b_3) b_3 - b_1^2 b_4 > 0,\ b_4 > 0$$

Example 5.4 By means of the Routh criterion, determine the stability of the systems represented by the following characteristic equations. For systems found to be unstable, determine the number of roots of the characteristic equation in the right-half of the s-plane.

(a) $s^5 + s^4 + 24s^3 + 48s^2 - 25s - 5 = 0$

(b) $s^4 + 2s^3 + 10s^2 + 8s + 3 = 0$

(c) $s^4 + 4s^3 + s^2 + 8s + 1 = 0$

(d) $s^5 + s^4 + 6s^3 + 12s^2 + 18s + 6 = 0$

(e) $s^6 + 2s^5 + 8s^4 + 15s^3 + 20s^2 + 16s + 16 = 0$

Solution:

(a) In the given characteristic equation

$$s^5 + s^4 + 24s^3 + 48s^2 - 25s - 5 = 0$$

the coefficient of s and the constant term are negative. Hence the necessary condition for stability is not satisfied and so the system is unstable. There is no need to formulate the Routh table to check for stability. However, to determine the location of the roots in the s-plane, formulate the Routh table as shown below.

s^5	1	24	–25
s^4	1	48	–5
s^3	$\frac{1\times 24-1\times 48}{1}=-24$	$\frac{1\times(-25)-1\times(-5)}{1}=-20$	
s^2	$\frac{-24\times 48-1\times(-20)}{-24}=47.16$	$\frac{-24\times(-5)-1\times 0}{-24}=-5$	
s^1	$\frac{47.16\times(-20)-(-24)\times(-5)}{47.16}=-22.54$		
s^0	$\frac{-22.54\times(-5)-47.16\times 0}{-22.54}=-5$		

There are three sign changes (s^4 row to s^3 row, s^3 row to s^2 row and s^2 row to s^1 row) in the elements of the first column of the Routh array. Hence, there are three roots of the characteristic equation in the right-half of the s-plane.

(b) $s^4 + 2s^3 + 10s^2 + 8s + 3 = 0$

The Routh table is formed as follows:

s^4	1	10	3
s^3	2	8	
s^2	$\frac{2\times 10-1\times 8}{2}=6$	$\frac{2\times 3-1\times 0}{2}=3$	
s^1	$\frac{6\times 8-2\times 3}{6}=7$	$\frac{6\times 0-2\times 0}{6}=0$	
s^0	$\frac{7\times 3-6\times 0}{7}=3$		

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To find the other roots, factorize the characteristic equation

$$
\begin{aligned}
s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 &= (s^2 + 1)(s^3 + 4s^2 + 7s + 4) \\
&= (s^2 + 1)(s + 1)(s^2 + 3s + 4) \\
&= (s^2 + 1)(s + 1)\left(s + 1.5j\sqrt{\frac{7}{4}}\right)\left(s + 1 - j\sqrt{\frac{7}{4}}\right)
\end{aligned}
$$

Alternative method: In fact, the moment difficulty 2 arises (i.e., a row of zeros exists), we can conclude that the system is unstable. The roots on the imaginary axis can be obtained by solving the auxiliary equation, i.e.

$$s^2 + 1 = 0$$

The characteristic equation is divided by the auxiliary equation

$$
\begin{array}{r|l}
 & s^3 + 4s^2 + 7s + 4 \\
\hline
s^2 + 1 & s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 \\
 & s^5 \qquad + s^3 \\
\hline
 & \quad 4s^4 + 7s^3 + 8s^2 + 7s + 4 \\
 & \quad 4s^4 \qquad + 4s^2 \\
\hline
 & \qquad 7s^3 + 4s^2 + 7s + 4 \\
 & \qquad 7s^3 \qquad + 7s \\
\hline
 & \qquad\quad + 4s^2 \qquad + 4 \\
 & \qquad\quad + 4s^2 \qquad + 4 \\
\hline
 & \qquad\qquad\qquad 0
\end{array}
$$

i.e. $\quad s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = (s^2 + 1)(s^3 + 4s^2 + 7s + 4)$

Apply Routh's test on $s^3 + 4s^2 + 7s + 4 = 0$ to check the location of the other roots.

s^3	1	7
s^2	4	4
s^1	6	0
s^0	4	

There is no change in the signs of the elements of the first column of the Routh array. Hence there is no root of the revised characteristic equation in the right-half of the s-plane. Since there is one pair of poles on the imaginary axis, the system is marginally stable.

(c) In the given characteristic equation

$$s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

the coefficients of s^4, s^3, s^2, s and constant term are negative. Hence the necessary condition for stability is not satisfied, and so the system is unstable. There is no need to formulate the Routh table to test for stability. However, to determine the location of the roots of the characteristic equation in the s-plane, the Routh table is formulated as follows:

s^6	1	–2	–7	–4
s^5	1	–3	–4	
s^4	$\frac{1\times(-2)-1\times(-3)}{1}=1$	$\frac{1\times(-7)-1\times(-4)}{1}=-3$	–4	
s^3	$\frac{1\times(-3)-1\times(-3)}{1}=0$	$\frac{1\times(-4)-1\times(-4)}{1}=0$		
s^2				
s^1				
s^0				

There is a row of zeros (all the elements of s^3 row are zeros), i.e. difficulty 2 arises, which indicates that there are symmetrically located roots in the s-plane.

The auxiliary equation is

$$A(s) = s^4 - 3s^2 - 4 = 0$$

i.e. $$A(s) = (s^2 - 4)(s^2 + 1) = 0$$

i.e. $$A(s) = (s + 2)(s - 2)(s + j)(s - j)$$

That is, there is a pair of conjugate roots on the imaginary axis, and there is a pair of symmetrically located roots on the real axis.

To determine the other roots, factorize the characteristic equation

$$s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = (s^4 - 3s^2 - 4)\ (s^2 + s + 1)$$

$$s^2 + s + 1 = (s + 0.5 + j\sqrt{0.75})(s + 0.5 - j\sqrt{0.75})$$

Therefore, the roots of the characteristic equation are $s = +j1$, $s = -j1$, $s = +2$, $s = -2$, $s = -0.5 + j\sqrt{0.75}$ and $s = -0.5 - j\sqrt{0.75}$.

Example 5.8 Using the Routh criterion, check whether the system represented by the following characteristic equation is stable or not. Comment on the location of the roots. Determine the frequency of sustained oscillations if any.

$$s^4 + 2s^3 + 6s^2 + 8s + 8 = 0$$

Solution: The Routh table is formulated as follows:

s^4	1	6	8
s^3	2	8	

s^4	1	13	K
s^3	4	36	
s^2	$\dfrac{4\times 13-1\times 36}{4}=4$	$\dfrac{4\times K-1\times 0}{4}=K$	
s^1	$\dfrac{4\times 36-4\times K}{4}=36-K$	0	
s^0	$\dfrac{(36-K)K-4\times 0}{36-K}=K$		

For stability, all the elements in the first column of the Routh array must be positive. Therefore, from the s^0 row

$$K > 0$$

and from the s^1 row

$$36 - K > 0$$

i.e. $$K < 36$$

Since the open-loop gain K must be positive, the range of values of K for stability is

$$0 < K < 36$$

(c) The Routh table for the characteristic equation

$$s^4 + 20Ks^3 + 5s^2 + 10s + 15 = 0$$

is formed as follows:

s^4	1	5	15
s^3	$20K$	10	
s^2	$\dfrac{20K\times 5-1\times 10}{20K}$	$\dfrac{20K\times 15-1\times 0}{20K}=15$	
s^1	$\dfrac{\dfrac{(100K-10)\times 10}{20K}-20K\times 15}{\dfrac{100K-10}{20K}}=\dfrac{100K-10-600K^2}{10K-1}$		
s^0	15		

For stability, all the elements in the first column of the Routh array must be positive. Therefore, from the s^3 row,

$$20K > 0$$

i.e. $$K > 0$$

From the s^2 row

$$100K > 10$$

i.e. $$K > 0.1$$

From the s^1 row K must be complex.

Since K must be a real positive number, the third condition cannot be satisfied. So the system is always unstable.

Example 5.10 The characteristic equation of a feedback control system is given by

$$s^4 + 20s^3 + 15s^2 + 2s + K = 0$$

(a) Determine the range of values of K for the system to be stable.

(b) Can the system be marginally stable? If so, find the required value of K and the frequency of sustained oscillations.

Solution:

(a) The Routh table for the given characteristic equation is formed as follows:

s^4	1	15	K
s^3	20	2	
s^2	$\dfrac{20\times 15-1\times 2}{20}=14.9$	$\dfrac{20\times K-1\times 0}{20}=K$	
s^1	$\dfrac{14.9\times 2-20\times K}{14.9}$	0	
s^0	K		

For the system to be stable, all the elements in the first column of the Routh array must be positve. Therefore, from the s^0 row

$$K > 0$$

and from the s^1 row

$$2 \times 14.9 - 20K > 0$$

i.e. $$K < 1.49$$

So the range of values of K for stability is $0 < K < 1.49$.

(b) Yes, the system can be marginally stable, for the marginal value of K, i.e. for $2 \times 14.9 - 20K = 0$, i.e. for $K = 1.49$. To obtain the frequency of sustained oscillations, substitute this critical value of K in the auxiliary equation formed by using the coefficients of the row just above the row which gives the critical value of K and solve it, i.e.

i.e. $$A(s) = 14.9s^2 + K = 0$$

i.e. $$14.9s^2 + 1.49 = 0$$

$\therefore$ $$s = \pm j\sqrt{\frac{1.49}{14.9}} = \pm j\sqrt{0.1} = \pm j0.316$$

Therefore, the frequency of sustained oscillations is 0.316 rad/s.

Example 5.11 A unity feedback system has the following open-loop transfer function:

$$G(s)=\frac{Ks(3s+1)}{s^2+2s+3}$$

Discuss the stability of the system in terms of the parameter K.

Solution: Given

$$G(s)=\frac{Ks(3s+1)}{s^2+2s+3},\ H(s)=1$$

The characteristic equation is

$$1 + G(s)\,H(s) = 0$$

i.e.
$$1+\frac{Ks(3s+1)}{s^2+2s+3}=0$$

i.e.
$$s^2 + 2s + 3 + 3Ks^2 + Ks = 0$$

i.e.
$$s^2(3K + 1) + s(K + 2) + 3 = 0$$

This is a second-order system. For the second-order system to be stable, it is sufficient that all the coefficients of the characteristic equation must be real and positive. Therefore,

$$3K + 1 > 0$$

i.e.
$$K > -1/3$$

and
$$K + 2 > 0$$

i.e.
$$K > -2$$

Since K the open-loop gain is positive, the system is stable for all positive values of K.

Example 5.12 The open-loop transfer function of a closed-loop system with unity feedback is

$$G(s)=\frac{K}{(s+2)(s+4)(s^2+6s+25)}$$

By applying the Routh criterion, discuss the stability of the closed-loop system as a function of K. Determine the values of K which will cause sustained oscillations in the closed-loop system. What are the corresponding oscillation frequencies?

Solution: Given

$$G(s)=\frac{K}{(s+2)(s+4)(s^2+6s+25)},\ H(s)=1$$

The characteristic equation is

$$1+G(s)H(s)=1+\frac{K}{(s+2)(s+4)(s^2+6s+25)}=0$$

25. What is the necessary and sufficient condition for stability using the Routh method?

A. The necessary and sufficient condition for stability of a system using the Routh method is that, each term of the first column of the Routh array of its characteristic equation be positive if $a_0 > 0$. If this condition is not met, the system will be unstable and the number of sign changes of the terms of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right-half of the s-plane.

26. What conclusion can be drawn if the first element of any row of a Routh array is zero when that row has at least one non-zero element?

A. If the first element in any row of a Routh array is zero, when that row has at least one non-zero element, it can be concluded that the corresponding system is unstable. It indicates that there are roots either in the right-half of the s-plane or on the imaginary axis.

27. What conclusion can be made if there is a row of all zeros in the Routh array?

A. If there is a row of all zeros in the Routh array, it can be concluded that the corresponding system is unstable. It indicates that there are symmetrically located roots in the s-plane with respect to its origin (pair of real roots with opposite signs and / or pair(s) of conjugate roots on the imaginary axis and / or complex conjugate roots forming quadrates in the s-plane).

28. What is an auxiliary polynomial? What is its order?

A. The polynomial whose coefficients are the elements of the row just above the row of zeros in the Routh array is called an auxiliary polynomial. The order of the auxiliary polynomial is always even. It is a factor of the characteristic equation.

29. What information does the auxiliary polynomial give?

A. The auxiliary polynomial gives the number and location of root pairs of the characteristic equation which are symmetrically located in the s-plane.

30. What conclusion can be drawn about the stability of a system whenever difficulty 1 or difficulty 2 arises?

A. Whenever difficulty 1 or difficulty 2 arises, it can be immediately concluded that the system is unstable.

31. What is quadrantal symmetry?

A. Quadrantal symmetry means there are symmetrical roots in all four quadrants. This happens when we have complex conjugate roots forming the quadrates in the s-plane.

32. What is difficulty 1 in Routh method? What does it indicate? How is it overcome?

A. Difficulty 1 in the Routh method is that, the first element in any row of the Routh array is zero while the rest of the row has at least one non-zero element. It indicates that the system is unstable due to the presence of poles either on the imaginary axis or in the right half of s-plane. Whenever difficulty 1 arises, all the elements in the next row become infinite and Routh's test fails. Only to find the number of roots in the right-half of the s-plane, one should proceed with the formulation of the Routh array.

Difficulty 1 is overcome by replacing the first element '0' by a small positive number $\in$ and proceeding with the formulation of the Routh array; or replacing s in the characteristic equation by $1/z$ and forming the Routh table on the modified characteristic equation in z.

33. What is difficulty 2 in the Routh method? What does it indicate? How is it overcome?

A. Difficulty 2 in Routh method is that, all the elements in any row of the Routh array are zero. It indicates that the system is unstable due to the presence of symmetrically located roots with respect to the origin of the s-plane. Only to find the location of the roots one should proceed with the formulation of the Routh array.

Difficulty 2 may be overcome by replacing the row of zeros by the coefficients of the first derivative of the auxiliary equation formed by using the coefficients of the row just above the row of zeros and proceeding with the formulation of the Routh array.

34. What are the advantages and drawbacks of the Routh stability criterion?

A. The Routh stability criterion is an algebraic method to test the stability of control systems. It is very simple because only the coefficients of the characteristic equation are to be manipulated to determine the stability of a system. It can be used to determine the condition for stability of linear feedback systems, but the drawbacks are as follows:

(a) For systems of order more than 6 or 7, it becomes cumbersome.

(b) It cannot be applied if the coefficients of the characteristic equation are complex.

(c) It is useful to find out only the absolute stability of a system.

(d) It is very cumbersome to obtain the relative stability of a system using this method. It has to be repeatedly applied to determine the relative stability.

(e) For an unstable system, it can tell only how many roots of the characteristic equation are in the right-half of the s-plane, but it cannot give the exact location of the roots. Also it cannot tell whether the roots are real or complex.

35. How can you determine the relative stability of a system by using the Routh stability criterion?

A. The relative stability of a system can be determined quantitatively by finding the settling time of the dominant roots of its characteristic equation. The settling time is inversely proportional to the real part of the dominant roots. So the relative stability can be specified by requiring that all the roots of the characteristic equation be more negative than a certain value, i.e. all the roots must lie to the left of the lines $s = -s_1 (s_1 > 0)$. Modify the given characteristic equation by shifting the origin of the s-plane to $s = -\sigma$ by substituting $s = z - \sigma$ and apply the Routh criterion on the new characteristic equation. If the criterion is satisfied, it implies that all the roots of the original characteristic equation are more negative than $-\sigma$.

36. What does the presence of negative terms during the formulation of the Routh array mean?

A. The presence of negative terms during the formulation of the Routh array means that the characteristic equation has one or more roots in the right-half of the s-plane and hence the system is unstable.

REVIEW QUESTIONS

1. State and explain the Routh stability criterion.
2. Show that the BIBO stability is satisfied if the impulse response $g(t)$ is absolutely integrable.
3. What are the difficulties that may be encountered in the formulation of the Routh array? What do they indicate? How are they overcome?
4. Explain how relative stability of a system can be determined using the Routh criterion.

FILL IN THE BLANKS

1. The two notions of stability are essentially equivalent in __________ systems.
2. A free stable __________ system will be stable for any bounded input.
3. For a free stable __________ system, there is no guarantee that the output will be bounded for a particular bounded input.
4. A linear time-invariant system is stable, if the following two notions of system stability are satisfied:
   (a) __________.
   (b) __________.
5. Roughly speaking stability in a system implies that small changes in __________, __________, and __________ do not result in large changes in system output.
6. Stability is a __________ term, whereas relative stability is a __________ term.
7. If all the roots of the characteristic equation have __________, then the impulse response is bounded and the system is stable.
8. If any root of the characteristic equation has a __________, the impulse response is unbounded and the system is unstable.
9. The settling time of poles is __________ proportional to the real part of the poles.
10. A system is said to be absolutely stable with respect to a parameter of the system, if it is stable for __________.
11. A system is said to be conditionally stable with respect to a parameter of the system, if it is stable for __________.
12. The necessary as well as sufficient condition for the stability of __________ and __________ order systems is that all the coefficients of the characteristic equation be real and positive.
13. Stability is a very important characteristic of the __________ response of the system.
14. Whenever difficulty 1 or difficulty 2 arises in the formulation of the Routh array, it can be concluded that the system is __________.

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i.e. $$s = \frac{-12 \pm \sqrt{144 - 96}}{6} = -2 \pm \sqrt{\frac{48}{36}} = -2 \pm 1.15$$

Therefore, the break points are $s = -3.15$ and $s = -0.85$. Out of these two, $s = -0.85$ is the actual break point because the root locus exists there. $s = -3.15$ is not an actual break point because the root locus does not exist there, and so it can be ignored. The break angles at $s = -0.85$ are

$$\pm \frac{\pi}{r} = \pm \frac{180^\circ}{2} = \pm 90^\circ$$

7. There is no need to compute the angles of departure and arrival as there are no complex poles and zeros.
8. The point of intersection of the root locus with the imaginary axis, and the marginal value of K can be determined by applying the Routh criterion. The characteristic equation is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+2)(s+4)}$$

i.e. $$s^3 + 6s^2 + 8s + K = 0$$

The Routh table is as follows:

s^3	1	8
s^2	6	K
s^1	$\frac{48 - K}{6}$	0
s^0	K	

For stability, all the elements in the first column of the Routh array must be positive. Therefore,

$$K > 0$$

and $$48 - K > 0$$

i.e. $$K < 48$$

Therefore, the range of values of K for stability is

$$0 < K < 48$$

The marginal value of K for stability is $K_m = 48$. The frequency of sustained oscillations is given by the solution of the auxiliary equation.

$$6s^2 + K = 0$$

i.e. $$6s^2 + K_m = 0$$

i.e. $$6s^2 + 48 = 0$$

$\therefore$ $$s^2 = -8$$

or $$s = \pm j\sqrt{8} = \pm j2.828$$

Therefore, the frequency of sustained oscillations is $\omega = 2.828$ rad/s.

Alternative way: The characteristic equation is given by

$$s^3 + 6s^2 + 8s + K = 0$$

Substituting $s = j\omega$ in the above equation, we get

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

i.e. $$-j\omega^3 - 6\omega^2 + j8\omega + K = 0$$

Equating the imaginary part to zero, we get

$$-\omega^3 + 8\omega = 0$$

$\therefore$ $$\omega^2 = 8$$

or $$\omega = \sqrt{8} = 2.828 \text{ rad/s}$$

Equating the real part to zero, we get

$$K - 6\omega^2 = 0$$

$\therefore$ $$K = 6\omega^2 = 6 \times 8 = 48$$

The values of ω and K are the same as obtained earlier.

The complete root locus sketch is shown in Figure 6.12. The root locus has three branches. The branch starting at $s = -4$ travels to the left on the negative real axis and terminates at $s = -\infty$. The branch starting at $s = 0$ moves to the left, the branch starting at $s = -2$ moves to the right, both meet at $s = -0.85$ (break point) break away and enter the complex plane and cross the imaginary axis at $\omega = \pm 2.828$ with $K = 48$, cross over to the right-half of the s-plane and travel along the asymptotes at 60° and 300^0 and terminate at $s = \infty$.

For damped oscillatory response, the dominant poles must be complex conjugate of each other.

The value of K at the break point can be determined by using the magnitude condition. So,

$$K \text{ (at break point)} = 0.85 \times 1.15 \times 3.15 = 3.08$$

The difference between the values of K at the break point and at the point of intersection of the root locus with the imaginary axis gives the range of K for which the system has damped oscillatory response. From the graph it is

$$3.08 < K < 48$$

To find the value of K for $\xi = 0.5$,

$$\theta = \cos^{-1} \xi = \cos^{-1} 0.5 = 60°$$

Draw a line making an angle of $\theta = 60°$ with respect to the negative real axis. Let it intersect the root locus at A. Since there are no open-loop zeros, the product of phasor lengths drawn from A to all the open-loop poles gives the value of K at A. Therefore, K for $\xi = 0.5$ is

$$K = xyz = 1.3 \times 1.8 \times 3.5 = 8.19$$

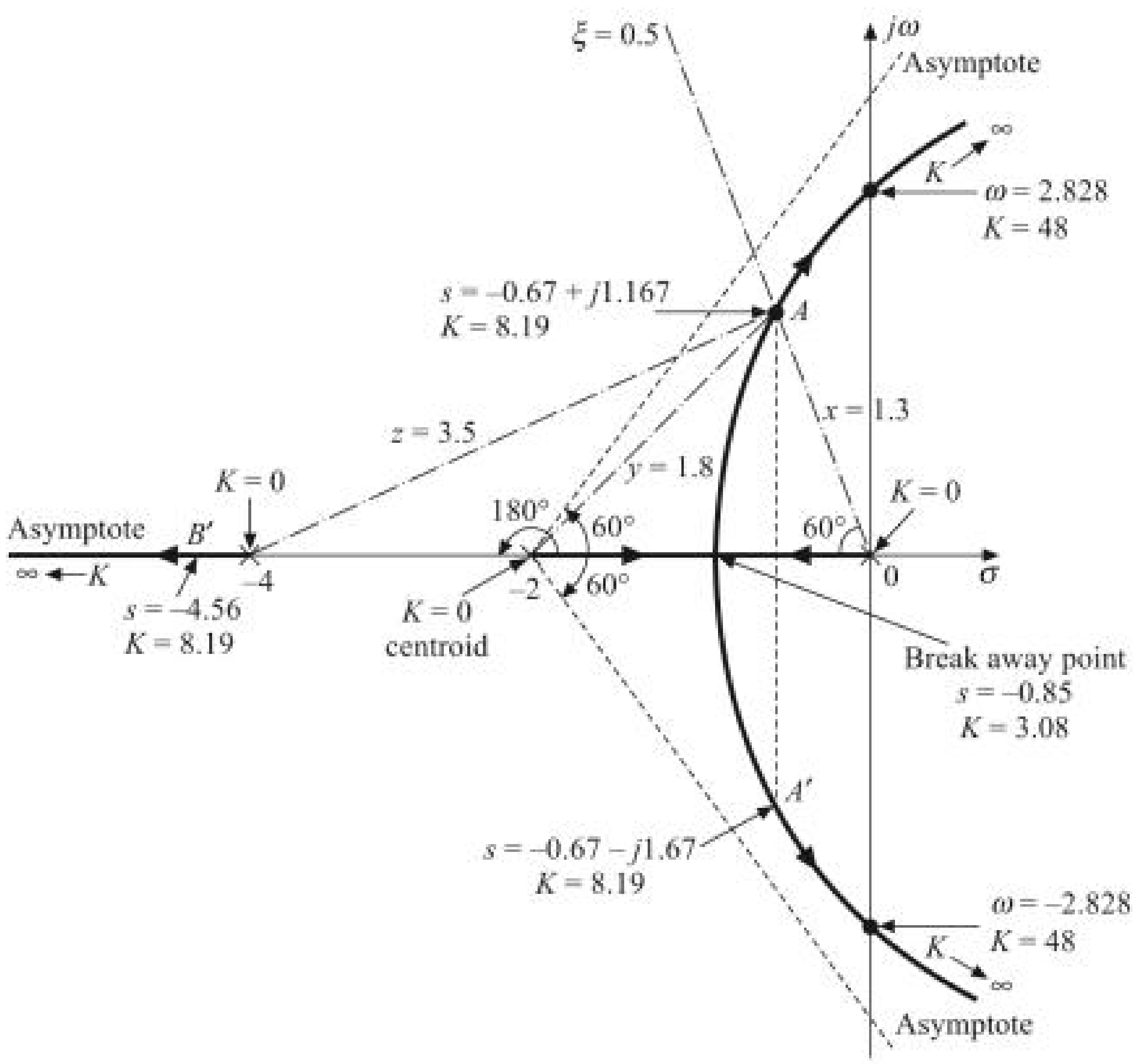

Figure 6.12 Example 6.5: Root locus.

To determine the closed-loop transfer function for $\xi = 0.5$, find K for $\xi = 0.5$ and locate the pair of complex poles at the point of intersection of constant ξ ($\xi = 0.5$) line with the root locus. From the root locus plot, it is $-0.67 + j1.16$. The second root is the conjugate of the first one. So it is $-0.67 - j1.16$. The third root will lie on the third branch of the root locus and can be determined by trial and error method (satisfying the magnitude condition). From the root locus plot it is $s = -4.56$ at B'. Therefore, the closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{8.19}{(s + 0.67 + j1.16)(s + 0.67 - j1.16)(s + 4.56)}$$

Example 6.6 A unity feedback system has an open-loop transfer function

$$G(s)H(s) = \frac{K}{s^2(s+2)}$$

(a) By sketching a root locus plot, show that the system is unstable for all values of K.

(b) Add a zero at $s = -a$ $(0 \le a < 2)$ and show that the addition of a zero stabilizes the system.

(c) If $a = 1$, sketch the root locus plot. Find the value of K which gives the greatest damping ratio for the oscillatory mode. Find also the value of this damping ratio and the corresponding undamped natural frequency.

Solution:

(a) For the given open-loop transfer function $G(s)H(s)$:

The open-loop poles are at $s = 0$, $s = 0$ and $s = -2$. Therefore, $n = 3$.
There are no finite open-loop zeros. Therefore, $m = 0$.
So, the number of branches of root locus $= n = 3$ and the number of asymptotes $= n - m = 3 - 0 = 3$.

The complete root locus is drawn as shown in Figure 6.13, as per the rules given below.

1. All the open-loop poles and zeros are on the real axis itself. So the root locus will be symmetrical with respect to the real axis.
2. The three branches of the root locus start at the open-loop poles $s = 0$, $s = 0$ and $s = -2$, where $K = 0$ and terminate at the open-loop zeros $s = \infty$, $s = \infty$ and $s = \infty$, where $K = \infty$.
3. There are three asymptotes and the angles of the asymptotes are

$$\theta_q = \frac{(2q+1)\pi}{n-m},\ q = 0, 1, 2$$

i.e.
$$\theta_0 = \frac{\pi}{3},\quad \theta_1 = \frac{3\pi}{3} = \pi,\quad \theta_2 = \frac{5\pi}{3}$$

4. The point of intersection of the asymptotes on the real axis (centroid) is given by

$$-\sigma = \frac{\text{sum of real parts of poles} - \text{sum of real parts of zeros}}{\text{number of poles} - \text{number of zeros}} = \frac{(0+0-2)-(0)}{3-0} = -0.666$$

5. The root locus exists on the real axis to the left of $s = -2$.
6. The break points are given by the solution of $\frac{dK}{ds} = 0$.

$$|G(s)H(s)| = \left|\frac{K}{s^2(s+2)}\right| = 1$$

$\therefore$
$$K = s^2(s + 2)$$

$$\frac{d}{ds}[s^2(s + 2)] = 3s^2 + 4s = 0$$

Therefore, the break points are at $s = 0$ and $s = -4/3$.
Out of these two, $s = 0$ is the actual break point. $s = -4/3$ is not an actual break point because the root locus does not exist at that point and hence can be ignored.
The break angles at $s = 0$ are

$$\pm\frac{\pi}{r} = \pm\frac{180^\circ}{2} = \pm 90^\circ$$

7. There are no complex poles and zeros. Therefore, there is no need to compute the angles of departure and arrival.
8. The point of intersection of the root locus with the imaginary axis and the critical value of K are obtained by using the Routh criterion. The characteristic equation is

$$1 + G(s)H(s) = 0$$

i.e.

$$1+\frac{K}{s^2(s+2)} = s^3 + 2s^2 + K = 0$$

In the characteristic equation, the coefficient of $s = 0$. Hence, the necessary condition for stability is not satisfied and so the system is unstable for all values of K.
The complete root locus sketch is shown in Figure 6.13. There are three branches of the root locus. The branch starting at $s = -2$, travels to the left on the negative real axis and terminates at $s = -\infty$. The other two branches start at $s = 0$, break away at $s = 0$ itself and remain in the right-half of the s-plane only and terminate at $s = \infty$. So the system is always unstable.

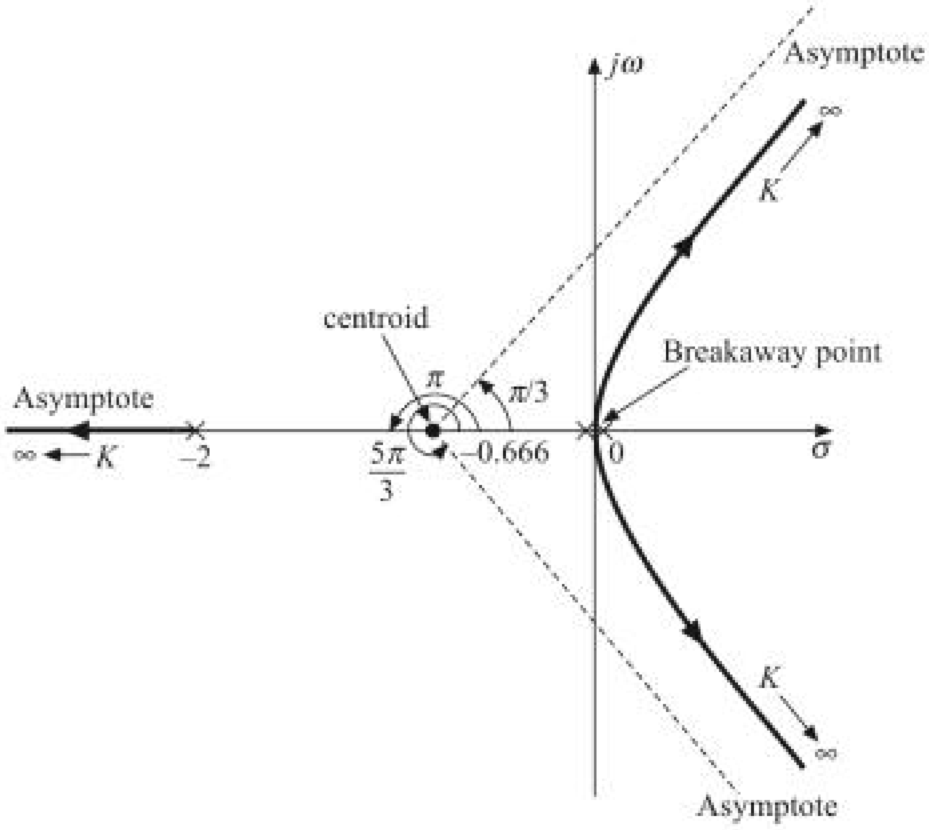

Figure 6.13 Example 6.6(a): Root locus.

(b) When a zero is added at $s = -a$ $(0 \le a < 2)$, the open-loop transfer function is

$$G(s)H(s) = \frac{K(s+a)}{s^2(s+2)}$$

Therefore, $n = 3$, and $m = 1$ and $n - m = 3 - 1 = 2$.

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6.9 Draw the complete root locus for a system with

$$G(s)H(s)=\frac{K(s+0.5)(s+2)}{s(s+1)(s-1)}$$

6.10 Draw the complete root locus for a system with

$$G(s)H(s)=\frac{2K(s+5)}{s^2(s^2+2s+2)}$$

6.11 Draw the complete root locus for a system with

$$G(s)H(s)=\frac{K(s+12)}{s^2(s+20)}$$

6.12 Sketch the root locus for the system with

$$G(s)H(s)=\frac{K}{s(s^2+6s+13)}$$

6.13 Sketch the root locus for the system with

$$G(s)H(s)=\frac{K(s+2)(s+3)}{(s+1)(s-1)}$$

Show that part of the root locus is a circle.

6.14 Sketch the root locus of the unity feedback system with open-loop transfer function

$$G(s)=\frac{K(s+2)}{(s+1)^2}$$

Show that part of the root locus is a circle. Find $\xi_{\min}$ and the corresponding value of K. Also find the transfer function for this $\xi_{\min}$. Determine the range of values of K for which the system is (a) overdamped, (b) critically damped and (c) underdamped.

6.15 A unity feedback control system has an open-loop transfer function

$$G(s)=\frac{K}{s(s^2+4s+13)}$$

Sketch the complete root locus of the system.

6.16 Sketch the root locus for the control system with

$$G(s)H(s)=\frac{K(s+4)}{s^2+4s+8}$$

Show that part of the root locus is a circle.

6.17 Sketch the root locus of the open-loop transfer function

$$G(s)H(s)=\frac{K(s^2+6s+25)}{s(s+1)(s+2)}$$

as K varies from zero to infinity.

7

Frequency Response Analysis

7.1 INTRODUCTION

Earlier, we have discussed that there are several standard test signals used to study the performance of control systems (viz. impulse, step, ramp, parabolic, and sinusoidal). Out of these, the sinusoidal signal is the most important one. Most of the signals available in nature are of sinusoidal type. The response of a system may be time response or frequency response. For time response, the input may be impulse, step, ramp or parabolic. For frequency response, the input must be sinusoidal. Consider a linear system, with a sinusoidal input.

$$r(t) = A \sin \omega t$$

Under steady-state, the system output as well as the signals at all other points in the system are sinusoidal. The steady-state out may be written as

$$c(t) = B \sin (\omega t + \phi)$$

The magnitude and phase relationship between the sinusoidal input and the steady-state output of a system is termed the *frequency response*. In linear time-invariant systems, the frequency response is independent of the amplitude and phase of the input signal.

The frequency response test on a system or a component is normally performed by keeping the amplitude A fixed and determining B and ϕ for a suitable range of frequencies.

The advantages of frequency response analysis are as follows:

1. Signal generators and precise measuring instruments are readily available for various ranges of frequencies and amplitudes.
2. The ease and accuracy of measurements is another advantage of the frequency response method.
3. Wherever it is not possible to obtain the form of the transfer function of a system through analytical techniques, the necessary information to compute its transfer function can be extracted by performing the frequency response test on the system. The step

response test can also be performed easily, but the extraction of transfer function from the step response data is quite a laborious procedure.

4. The design and parameter adjustment of the open-loop transfer function of a system for a specified closed-loop performance is carried out some what more easily in frequency-domain (i.e. through frequency response) than in time-domain (i.e. through time response).
5. The effects of noise disturbance and parameter variations are relatively easy to visualize and assess through frequency response.
6. If necessary, the transient response of a system can be obtained from its frequency response through the Fourier integral. This correlation between time and frequency response is quite tedious to compute except for first- and second-order systems. Usually, for the sake of simplicity and ease of analysis, the correlation between frequency and time response of a second-order system is employed as an approximation to higher-order systems as well. Through the use of this approximate correlation, a satisfactory transient response for higher-order systems can be achieved by the adjustment of their frequency response.

The drawbacks of frequency response method are as follows:

1. For systems with very large time constants, the frequency response test is cumbersome to perform, as the time required for the output to reach the steady-state for each frequency of the test signal is excessively long. Therefore, the frequency response test is not recommended for systems with very large time constants.
2. The frequency response test obviously cannot be performed on non-interruptable systems. Under such circumstances, a single shot test (step or impulse) is more convenient even though the computation of the transfer function from it gets involved.

An interesting and revealing comparison of frequency and time-domain approaches is based on the relative stability studies of feedback systems. The Routh stability criterion is a time-domain approach, which establishes with relative ease the stability of a system, but its adoption to determine the relative stability is involved and requires repeated application of the criterion. The root locus method is a very powerful time-domain approach as it reveals not only stability, but also the actual time response of the system. On the other hand, the Nyquist criterion is a powerful frequency-domain method of extracting the information regarding stability as well as relative stability of a system without the need to evaluate the roots of the characteristic equation.

The frequency response is easily evaluated from the sinusoidal transfer function which can be obtained simply by replacing s with $j\omega$ in the system transfer function $T(s)$. The transfer function $T(j\omega)$ thus obtained, is a complex function of frequency and has both magnitude and phase angle. These characteristics are convenienently represented by graphical plots.

7.2 CORRELATION BETWEEN TIME AND FREQUENCY RESPONSE

The correlation between time and frequency response has an explicit form only for systems of first- and second-order.

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8 Nyquist Plot

8.1 INTRODUCTION

The Nyquist stability criterion relates the location of the roots of the characteristic equation to the open-loop frequency response of the system. In this, the computation of closed-loop poles is not necessary to determine the stability of the system and the stability study can be carried out graphically from the open-loop frequency response. Therefore, experimentally determined open-loop frequency response can be used directly for the study of stability, when the feedback path is closed.

The Nyquist criterion has the following features that make it an alternative method that is attractive for the analysis and design of control systems.

1. In addition to providing information on absolute and relative stabilities of a system, it can also indicate the degree of instability of an unstable system. It also gives indication on how the system stability may be improved, if needed.
2. The Nyquist plot of $G(s)H(s)$ is very easy to obtain especially with the aid of a computer.
3. The Nyquist plot of $G(s)H(s)$ gives information on the frequency-domain characteristics such as M_r, ω_r, BW, and others with ease.
4. The Nyquist plot is useful for systems with pure time delay that cannot be treated with the Routh criterion and are difficult to analyze with the root locus method.

Stability problem: The Nyquist criterion represents a method of determining the location of the characteristic equation roots with respect to the left-half and the right-half of the s-plane. Unlike the root locus method, the Nyquist criterion does not give the exact location of the characteristic equation roots. The Nyquist stability criterion is based on a theorem of complex variables due to Cauchy, commonly known as the *principle of argument*.

Stability conditions: We define two types of stability with respect to the system configuration.

Open-loop stability: A system is said to be open-loop stable if the poles of the loop transfer function $G(s)H(s)$ are all in the left-half of the s-plane. For a single loop system, this is equivalent to the system being stable when the loop is opened at any point.

Closed-loop stability: A system is said to be closed-loop stable, or simply stable, if the poles of the closed-loop transfer function, i.e., the zeros of $1 + G(s)H(s)$ are all in the left-half of the s-plane. Exceptions to the above definitions are systems with poles or zeros intentionally placed at $s = 0$.

8.2 PRINCIPLE OF ARGUMENT

Consider a function $q(s)$ that can be expressed as a quotient of two polynomials. Each polynomial may be assumed to be known in the form of product of linear factors indicated as follows:

$$q(s) = \frac{(s-\alpha_1)(s-\alpha_2)\ldots(s-\alpha_m)}{(s-\beta_1)(s-\beta_2)\ldots(s-\beta_n)} \tag{8.1}$$

Let s be a complex variable, represented by $s = \sigma + j\omega$ on the complex s-plane. Then the function $q(s)$ is also complex (being a dependent variable) and may be defined as $q(s) = u + jv$ and represented on the complex $q(s)$ plane with coordinates u and v. Equation (8.1) indicates that for every point s in the s-plane at which $q(s)$ is analytic [A function $q(s)$ is said to be analytic in the s-plane provided the function and all its derivatives exist. The points in the s-plane where the function (or its derivatives) does not exist are called singular points. The poles of a function are singular points.], we can find a corresponding point $q(s)$ in the $q(s)$-plane. Alternatively, it can be stated that the function $q(s)$ maps the points in the s-plane into the $q(s)$-plane. Since any number of points in the s-plane, can be mapped into the $q(s)$-plane, it follows that for a contour in the s-plane which does not go through any singular point, there corresponds a contour in the $q(s)$-plane as shown in Figure 8.1. The region to the right of the closed contour is considered enclosed by the contour when the contour is traversed in the clockwise direction. Thus the shaded area in Figure 8.1(a) is enclosed by the closed contour.

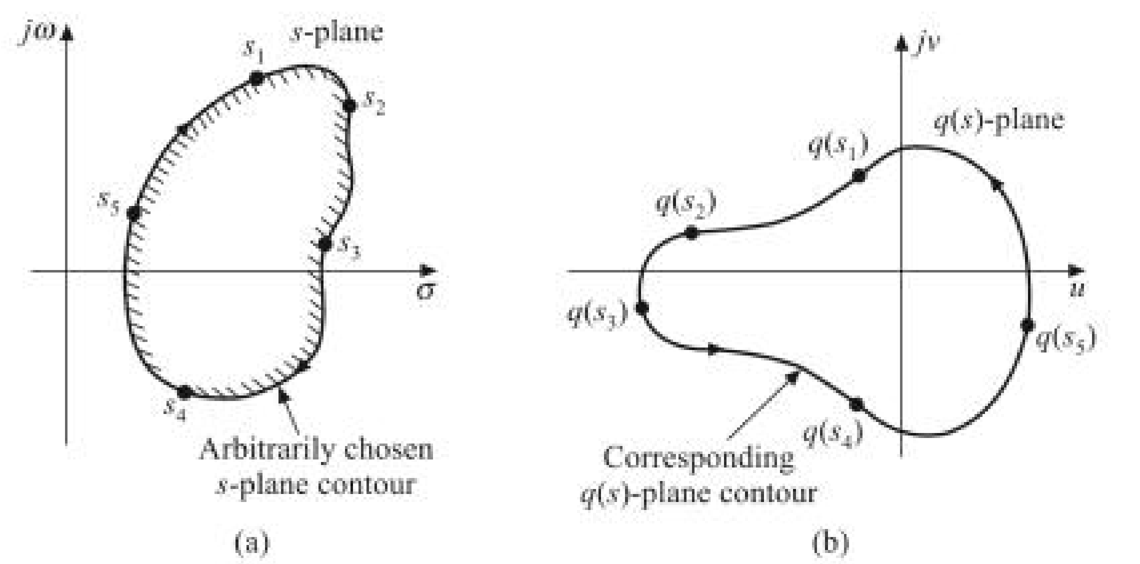

Figure 8.1 Arbitrarily chosen s-plane contour which does not go through singular points and the corresponding $q(s)$-plane contour.

While developing the Nyquist criterion, we are not interested in the exact shape of the $q(s)$-plane contour. An important fact that concerns us is the encirclement of the origin by the $q(s)$-plane contour. To investigate this, consider an s-plane contour shown in Figure 8.2(a) which encloses only one of the zeros of $q(s)$, say $s = \alpha_1$, while all the poles and remaining zeros are distributed in the s-plane outside the contour. As discussed above, for any non-singular point s in the s-plane contour, there corresponds a point $q(s)$ on the $q(s)$ plane contour. The point $q(s)$ is given by

$$|q(s)| = \frac{|s-\alpha_1||s-\alpha_2|\ldots}{|s-\beta_1||s-\beta_2|\ldots}$$

$$\angle q(s) = \angle(s-\alpha_1) + \angle(s-\alpha_2) + \ldots - \angle(s-\beta_1) - \angle(s-\beta_2) - \ldots$$

From Figure 8.2(a) it is found that as the point s follows the prescribed path (i.e. clockwise direction) on the s-plane contour, eventually returning to the starting point, the phasor $(s - \alpha_1)$ generates a net angle of -2π, while all other phasors generate zero net angles. Therefore, the $q(s)$-phasor undergoes a net phase change of -2π. This implies that the tip of the $q(s)$-phasor must describe a closed contour about the origin of the $q(s)$-plane in the clockwise direction. As said earlier, the exact shape of the closed contour in the $q(s)$-plane is not of interest to us, but it is sufficient for us to observe that this contour encircles the origin once.

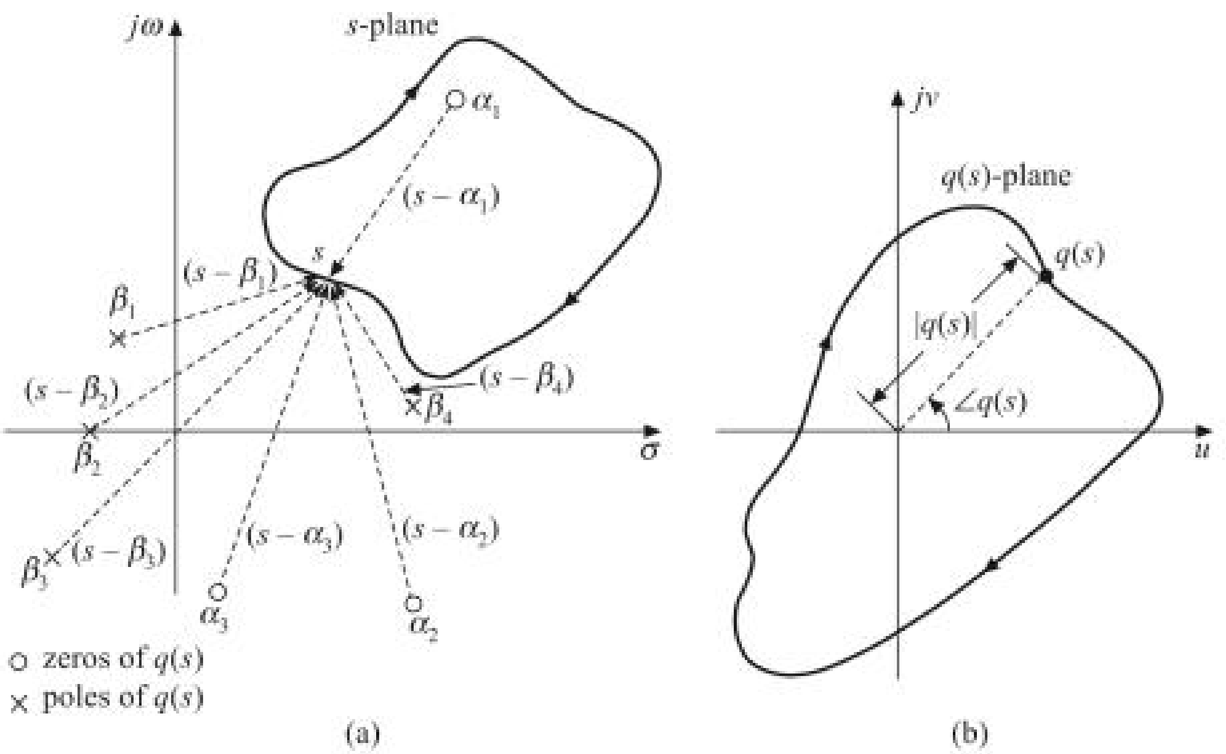

Figure 8.2 An s-plane contour enclosing a zero of $q(s)$ and the corresponding $q(s)$-plane contour.

If the contour in the s-plane does not enclose any zero or pole, the corresponding contour in the $q(s)$-plane then will not encircle the origin. If the s-plane contour encloses two zeros, say at $s = \alpha_1$ and $s = \alpha_2$, the $q(s)$-plane contour encircles the origin twice in the clockwise direction as shown in Figure 8.3(b). Generalizing, we can say that for each zero of $q(s)$ enclosed by the s-plane contour, the corresponding $q(s)$-plane contour encircles the origin once in the clockwise direction.

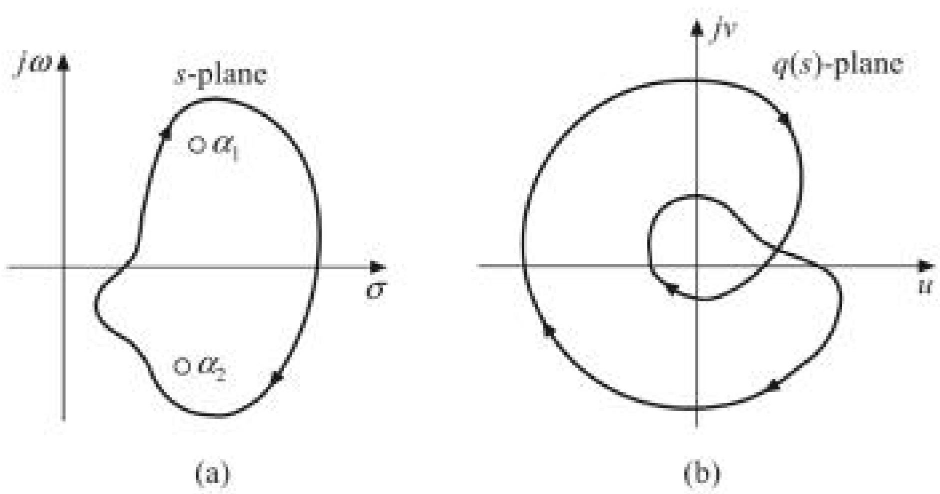

Figure 8.3 An s-plane contour enclosing two zeros of $q(s)$ and the corresponding $q(s)$-plane contour.

If the s-plane contour encloses a pole (at $s = \beta_1$) of $q(s)$, the phasor $(s - \beta_1)$ generates an angle of -2π as s traverses the prescribed path. Since the pole term $(s - \beta_1)$ is in the denominator of $q(s)$, the $q(s)$-plane contour experiences an angle change of $+2\pi$, which means one counterclockwise encirclement of the origin. This argument holds for all other poles of $q(s)$. If the s-plane contour encloses two poles say at $s = \beta_1$ and $s = \beta_2$, the $q(s)$-plane contour encircles the origin twice in the counterclockwise direction as shown in Figure 8.4.

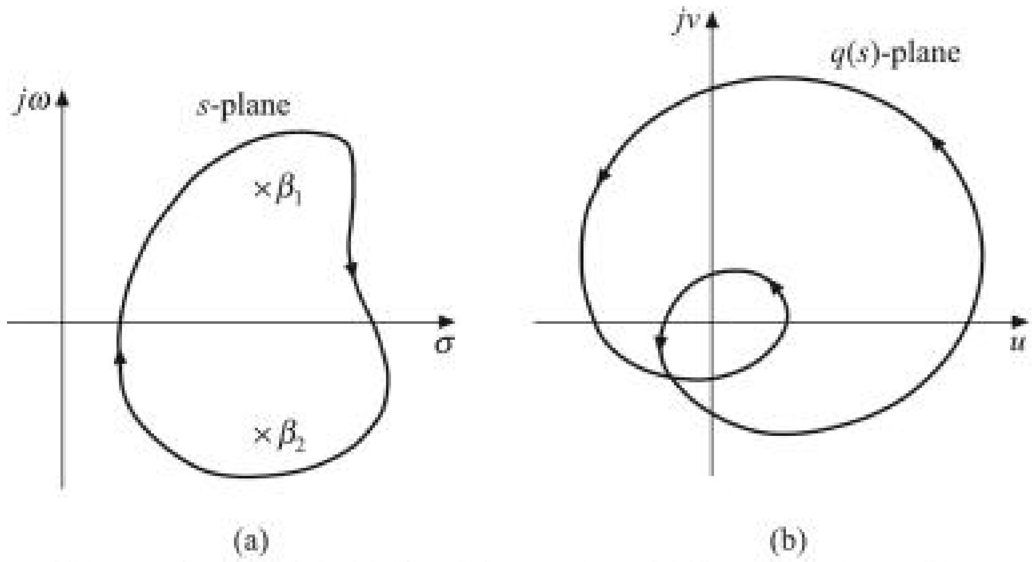

Figure 8.4 Mapping of the s-plane contour which encloses two poles.

Thus, if there are P poles and Z zeros of $q(s)$ enclosed by the s-plane contour, then the corresponding $q(s)$-plane contour must encircle the origin Z times in the clockwise direction and P times in the counterclockwise direction, resulting in a net encirclement of the origin, $(P - Z)$ times in the counterclockwise direction. For example, in case of two zeros and four poles enclosed by the s-plane contour, the net encirclement of the origin by the $q(s)$-plane contour is $2\pi\ (4 - 2) = 4\pi$ rad, i.e. two counterclockwise revolutions as shown in Figure 8.5(a) and Figure 8.5(b). This relation between the enclosure of poles and zeros of $q(s)$ by the s-plane contour and the encirclements of the origin by the $q(s)$-plane contour is commonly known as the *principle of argument*.

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The following systems are not completely state controllable:

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix}+\begin{bmatrix}3\\ 0\end{bmatrix}u$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix}=\begin{bmatrix}-2 & 1 & 0\\ 0 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}+\begin{bmatrix}3 & 2\\ 0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\\ \dot{z}_4\\ \dot{z}_5\end{bmatrix}=\begin{bmatrix}-3 & 1 & 0 & 0 & 0\\ 0 & -3 & 1 & 0 & 0\\ 0 & 0 & -3 & 0 & 0\\ 0 & 0 & 0 & -4 & 1\\ 0 & 0 & 0 & 0 & -4\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\\ z_4\\ z_5\end{bmatrix}+\begin{bmatrix}3\\ 2\\ 1\\ 4\\ 0\end{bmatrix}u$$

10.10.2 Condition for Complete State Controllability in the *s*-Plane

The condition for complete state controllability can be stated in terms of transfer functions or transfer matrices.

A necessary and sufficient condition for complete state controllability is that no cancellation occurs in the transfer function or transfer matrix. If cancellation occurs, the system can not be controlled in the direction of the cancelled mode.

10.10.3 Output Controllability

In the practical design of a control system, we may want to control the output rather than the state of the system. Complete state controllability is neither necessary nor sufficient for controlling the output of the system. For this reason, it is desirable to define separately complete output controllability.

The system described by

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)$$

is said to be completely output controllable, if it is possible to construct an unconstrained controlled vector $\mathbf{u}(t)$ that will transfer any given initial output $y(t_0)$ to any final output $y(t_f)$ in a finite time interval $t_0 \le t \le t_f$.

The condition for complete output controllability is – the system described by the state model

$$\dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t)=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)$$

is completely output controllable, if and only if the $m \times (n+1)$ matrix

$$[\mathbf{CB} \quad \mathbf{CAB} \quad \mathbf{CA^2B} \; \ldots \; \mathbf{CA^{n-1}B} \quad \mathbf{D}]$$

is of rank m, where m is the number of outputs.

The presence of the **Du** term in the output equation always helps to establish controllability.

Uncontrollable system: An uncontrollable system has a sub-system that is physically disconnected from the input.

Example 10.35 Determine whether the system described by the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

is completely state controllable.

Solution: First transform this state equation into canonical form. The characteristic equation is given by

$$|\lambda \mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 0 & 2 & \lambda+3 \end{vmatrix} = \lambda^3 + 3\lambda^2 + 2\lambda + 2 = \lambda(\lambda+1)(\lambda+2) = 0$$

Therefore, the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = -1$ and $\lambda_3 = -2$. Since the eigenvalues are distinct, choosing Vander Monde matrix as modal matrix, we have

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \frac{\begin{bmatrix} -2 & 0 & 0 \\ -3 & 4 & -1 \\ -1 & 2 & -1 \end{bmatrix}^T}{-2} = \frac{\begin{bmatrix} -2 & -3 & -1 \\ 0 & 4 & 2 \\ 0 & -1 & -1 \end{bmatrix}}{-2}$$

$$\therefore \qquad \overline{\mathbf{B}} = \mathbf{M}^{-1}\mathbf{B} = \frac{\begin{bmatrix} -2 & -3 & -1 \\ 0 & 4 & 2 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}{-2} = \frac{\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}}{-2} = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

$$\mathbf{M}^{-1}\mathbf{AM} = -\frac{1}{2}\begin{bmatrix} -2 & -3 & -1 \\ 0 & 4 & 2 \\ 0 & -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution: Given the matrices **A** and **B**

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\mathbf{A^2B} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix}$$

The controllability matrix is

$$\mathbf{S} = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A^2B}] = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & 0 \\ 1 & 3 & 10 \end{bmatrix}$$

which is singular (the rank $r = 1$). Thus the system is not completely state controllable.

For the given matrices **A**, **B**, and **C**

$$\mathbf{CB} = [1 \quad 1 \quad 0] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\mathbf{CAB} = [1 \quad 1 \quad 0] \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & -4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -1$$

$$\mathbf{CA^2B} = [1 \quad 1 \quad 0] \begin{bmatrix} 4 \\ 0 \\ 10 \end{bmatrix} = 4$$

The rank of $[\mathbf{CB} \quad \mathbf{CAB} \quad \mathbf{CA^2B}] = [0 \quad -1 \quad 4]$ is 1 ($= m$).

Thus, the system is completely output controllable.

10.11 OBSERVABILITY OF LINEAR SYSTEMS

A system is completely observable, if every state variable of the system affects some of the outputs. In other words, it is often desirable to obtain information on the state variables from the measurements of the outputs and the inputs. If any one of the states cannot be observed from the measurements of the output, the state is said to be unobservable and the system is not completely observable or simply unobservable. Figure 10.33 shows the state diagram of a linear

32. How do you prove the invariance of eigenvalues?

A. The invariance of the eigenvalues under a linear transformation can be proved by showing that the characteristic polynomials $|\lambda\mathbf{I} - \mathbf{A}|$ and $|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}|$ are identical.

33. When do you say that the state model is stable?

A. The state model is said to be stable if all its eigenvalues have negative real parts.

34. What is the relation between the eigenvalues of the state model and the poles of the system transfer function?

A. The eigenvalues of the state model and the poles of the system transfer function are the same.

35. Under linear transformation of a state model, which parameters of the system are invariant?

A. In linear (similarity) transformation of the state model, the characteristic equation, eigenvalues, eigenvectors and the transfer function are all preserved, i.e. they are all invariant under a linear transformation.

36. How can you obtain the transfer function of a system from its state model?

A. The transfer function of a system can be obtained from its state model by using the formula

$$\frac{C(s)}{R(s)} = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\,\mathbf{B} + \mathbf{D}$$

37. Given its state model, how can you determine the stability of a system?

A. Since the eigenvalues of the system matrix **A** are the same as the roots of the characteristic equation which are nothing but the poles of the closed-loop transfer function, the stability of a system represented by its state model can be determined by determining the location of the eigenvalues.

38. When do you say that the system is completely state controllable?

A. A system is said to be completely state controllable at time t_0, if it is possible by means of an un constrained control vector $\mathbf{u}(t)$ to transfer the system from an initial state $\mathbf{x}(t_0)$ to any other desired state $\mathbf{x}(t)$ in a finite interval of time.

39. When do you say that the system is completely output controllable?

A. A system is said to be completely output controllable, if it is possible to construct an unconstrained control vector $\mathbf{u}(t)$ that will transfer any given initial output $\mathbf{y}(t_0)$ to a final output $\mathbf{y}(t_f)$ in a finite time interval $t_0 \leq t \leq t_f$.

40. When do you say that the system is observable?

A. A system is said to be observable at time t_0, if with the system in state $\mathbf{x}(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval.

41. Who introduced the concepts of controllability and observability?

A. The concepts of controllability and observability were introduced by Kalman.

42. What is the necessary and sufficient condition for state controllability of a system?

A. The necessary and sufficient condition for complete state controllability of a system as per Kalman is that the rank of the controllability matrix $\mathbf{S}_c = [\mathbf{B}\ \ \mathbf{AB}\ \ \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}]$ should be n.

43. What is the necessary and sufficient condition for output controllability of the system?

A. The necessary and sufficient condition for output controllability of a system is that the rank of the matrix $[\mathbf{CB}\ \ \mathbf{CAB} \ldots \mathbf{CA}^{n-1}\mathbf{B}\ \ \mathbf{D}]$ should be m.

44. What is the necessary and sufficient condition for observability of a system?

A. The necessary and sufficient condition for complete state observability of a system as per Kalman is that, the rank of the observability matrix $[\mathbf{C}^T\ \ \mathbf{A}^T\mathbf{C}^T \ldots (\mathbf{A}^T)^{n-1}\mathbf{C}^T]$ should be n.

45. State the duality property.

A. Duality property is that (a) the pair (**AB**) is controllable implies that the pair $(\mathbf{A}^T\mathbf{B}^T)$ is observable and (b) the pair **AC** is observable implies that the pair $(\mathbf{A}^T\mathbf{C}^T)$ is controllable.

46. What is the implication of pole-zero cancellation in a transfer function on controllability and observability?

A. If the input-output transfer function of a linear system has pole-zero cancellation, the system will be either uncontrollable or unobservable or both depending on how the state variables are defined. If the input-output transfer function of a linear system does not have pole-zero cancellation, we are assured that it is a controllable and observable system, no mater how the state variable model is defined.

47. When is the system always observable?

A. A system is always observable if it can be described in observable phase variable form.

48. When is the system always controllable?

A. A system is always controllable if it can be described in controllable canonical form.

49. What do you mean by an uncontrollable system?

A. An uncontrollable system is a system which has a sub-system that is physically disconnected from the input.

REVIEW QUESTIONS

1. Discuss the methods of decomposition of transfer functions.
2. Derive an expression for the solution of a homogeneous state equation.
3. Derive an expression for the solution of a non-homogeneous state equation.
4. Show that the eigenvalues are invariant under a linear transformation.
5. Obtain the transfer function of a system from its state model.

FILL IN THE BLANKS

1. The two basic approaches to the analysis and design of control systems are __________ and __________.
2. The transfer function appraoach is also called the __________ or __________ and the state variable approach is called the __________.
3. The transfer function approach is applicable only to __________ whereas the state variable approach can handle __________ as well as __________, __________ as well as __________, and __________ as well as __________ systems.
4. The transfer function approach __________ the initial conditions whereas the state variable approach __________ the initial conditions.
5. The transfer function approach is a __________ approach whereas state variable approach is a __________ approach.
6. Classical design methods are based on __________ procedure.
7. The state variables of a system are not __________.
8. The __________ of a system is not unique, whereas the __________ of a system is unique.
9. The number of elements in the state vector is refered to as the __________ of the system.
10. To obtain a state model for electrical networks __________ and the __________ are selected as the state variables.
11. In state variable formulation, the __________ are selected as the state variables.
12. In state-space analysis, three types of variables are involved in the modelling of dynamic systems. They are __________, __________ and __________.
13. The number of state variables to completely define the dynamics of the system is equal to the number of __________ involved in the system.
14. __________ in a continuous time control system serve as memory devices.
15. The state of a system refers to the __________, __________ and __________ conditions of the system.
16. $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ is known as the __________ of the system. **A** is called the __________ and **B** is called the __________.
17. $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$ is known as the __________ of the system. **C** is called the __________ and **D** is called the __________.
18. The state equations are a set of first-order differential equations, wherein the first derivatives of the state variables are expressed in terms of the __________ and the __________ of the system.
19. The output equations are the equations in which the outputs of the system are expressed in terms of the __________ and the __________ of the system.

OBJECTIVE TYPE QUESTIONS

1. The transfer function approach is applicable to
(a) only linear time-invariant systems
(b) linear time-invariant as well as time-varying systems
(c) linear as well as nonlinear systems
(d) all systems

2. The state variable approach is applicable to
(a) only linear time-invariant systems
(b) linear time-invariant as well as time-varying systems
(c) linear as well as nonlinear systems
(d) all systems

3. The number of state variables of a system is equal to
(a) the number of integrators present in the system
(b) the number of differentiators present in the system
(c) the sum of the number of integrators and differentiators present in the system
(d) none of these

4. Using state variables, an nth-order differential equation can be decomposed into
(a) n number of first-order differential equations
(b) $n/2$ number of first-order differential equations
(c) unlimited number of first-order differential equations
(d) none of these

5. $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ is called the
(a) system equation
(b) state equation
(c) state transition equation
(d) none of these

6. **A** is called the
(a) system matrix
(b) input matrix
(c) state matrix
(d) transmission matrix

7. **D** is called the
(a) system matrix
(b) transmission matrix
(c) transfer matrix
(d) output matrix

8. The number of canonical forms is
(a) 2
(b) 3
(c) 4
(d) 5

9. The number of ways in which STM can be computed is
(a) 2
(b) 3
(c) 5
(d) 6

10. $\phi(s)$ is called the

(a) state transition matrix (b) resolution matrix
(c) resolvent matrix (d) transfer matrix

11. An $n \times n$ matrix is said to be nonsingular if the rank of the matrix r is

(a) $r \neq n$ (b) $r = n$
(c) $r = n/2$ (d) $r = 2n$

12. Direct decomposition is applicable to transfer functions in which

(a) denominator is in factored form
(b) both numerator and denominator are in factored form
(c) both numerator and denominator are not in factored form
(d) all the above

13. The eigenvalues of the state model are the same as the

(a) open-loop poles
(b) closed-loop poles
(c) both the open-loop and closed-loop poles
(d) none of these

14. The concepts of controllability and observability were introduced by

(a) Gilbert (b) Gibson
(c) Kalman (d) none of these

Answers to Objective Type Questions

1. a **2.** d **3.** a **4.** a **5.** b **6.** a **7.** b **8.** d **9.** c **10.** c **11.** b **12.** c **13.** b **14.** c

PROBLEMS

10.1 A feedback system has a closed-loop transfer function

(a) $\dfrac{10(s+3)}{s(s+2)(s+4)}$ (b) $\dfrac{5}{(s+1)(s+2)^2}$ (c) $\dfrac{(s+1)(s+3)(s+5)}{(s+2)(s+4)(s+6)}$

Construct three different state models for this system and give block diagram representation for each state model.

10.2 Using parallel decomposition, construct state models for the systems represented by

(a) $\dfrac{3s+4}{(s+1)^3(s+2)^2(s+3)}$ (b) $\dfrac{1}{s(s+1)^2(s+2)}$ (c) $\dfrac{5(s+1)}{s(s+3)^2}$

10.3 Construct state models for the following differential equations

(a) $2\dddot{y} + 3\ddot{y} + 5\dot{y} + 2y = u$ (b) $\dddot{y} + 5\ddot{y} + 3\dot{y} + y(t) + \int_0^t y(\tau)\,d\tau = \int_0^t u(\tau)\,d\tau$

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